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On the Gauge/Gravity Correspondence and the Open/Closed String Duality

Paolo Di Vecchia [a], **Antonella Liccardo** [b], **Raffaele Marotta** [b] **and Franco Pezzella** [b]

[a] *NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*
e–mail: `divecchi@alf.nbi.dk`

[b] *Dipartimento di Scienze Fisiche, Università di Napoli and INFN, Sezione di Napoli*
Complesso Universitario Monte S. Angelo, ed. G - via Cintia - I-80126 Napoli, Italy
e–mail: `name.surname@na.infn.it`

Abstract

In this article we review the conditions for the validity of the *gauge/gravity correspondence* in both supersymmetric and non-supersymmetric string models. We start by reminding what happens in type IIB theory on the orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, where this correspondence beautifully works. In these cases, by performing a complete stringy calculation of the interaction among D3-branes, it has been shown that the fact that this correspondence works is a consequence of the open/closed duality and of the absence of threshold corrections. Then we review the construction of type 0 theories with their orbifolds and orientifolds having spectra free from both open and closed string tachyons and for such models we study the validity of the gauge/gravity correspondence, concluding that this is not a peculiarity of supersymmetric theories, but it may work also for non-supersymmetric models. Also in these cases, when it works, it is again a consequence of the open/closed string duality and of vanishing threshold corrections.

Contents

1 Introduction

A D-brane is characterized by the fundamental properties of being a solution of the low-energy string effective action that is given by supergravity (SUGRA) and of having open strings with their endpoints attached to its world-volume. In particular, the lightest open string excitations correspond to a gauge field and its supersymmetric partners if the theory is supersymmetric. These two complementary descriptions of a D-brane provide a powerful tool for deriving the *quantum* properties of the gauge theory living on the D-brane world-volume from the *classical* brane dynamics and viceversa. In particular, the fact that one can determine the gauge-theory quantities in terms of the supergravity solution goes under the name of *gauge/gravity correspondence*. This has allowed to derive properties of $\mathcal{N} = 4$ super Yang-Mills as one can see for example in Ref. [1] and, by the addition of a decoupling limit, also to formulate the Maldacena conjecture of the equivalence between $\mathcal{N} = 4$ super Yang-Mills and type IIB string theory compactified on $AdS_5 \otimes S^5$. [2]

Although it has not been possible to extend the Maldacena conjecture to non-conformal and less supersymmetric gauge theories, nevertheless a lot of apriori unexpected informations on these theories have been obtained from the gauge/gravity correspondence [1]. These more realistic gauge theories can be identified with the ones living on the world-volume of D5 branes wrapped either on a non-trivial 2-cycle of a non-compact Calabi-Yau space or on the "shrinking" 2-cycle located at the fixed point of an orbifold background. This second kind of wrapped D5 branes are called *fractional* D3 branes. They are branes stuck at the fixed point of an orbifold and are the ones that we will concentrate on in this review article because they admit an explicit stringy description. In this case, the role of the orbifold background is to reduce supersymmetry, while the one of fractional branes is to break conformal invariance. We must, however, stress that the conclusions we will draw from the fractional branes of an orbifold background seem also to be valid in the case of the wrapped branes described for instance by the Maldacena-Núñez [8, 9] and Klebanov-Strassler [10] classical solutions, although in these cases this cannot be checked because of the lack of an explicit stringy description.

In particular, it has been shown that the classical SUGRA solutions corresponding to those D-branes encode perturbative and non-perturbative properties of non-conformal and less supersymmetric gauge theories living on their world-volume as the chiral and scale anomalies and the superpotential. [11, 12, 13, 14, 15] It was of course expected that the perturbative properties could be derived from studying in string theory the gauge theory which lives on those D-branes by taking the field theory limit of one-loop open string annulus diagram using methods as those described for instance in Ref. [16]. But it came as a surprise that these properties were also encoded in the SUGRA solution, especially after the formulation of the Maldacena conjecture which relates the SUGRA approximation to the strong coupling regime of the gauge theory.

The explanation of this fact was given in Ref. [17] where it was shown, in the case of

[1]For general reviews on various approaches see for instance Ref.s [3, 4, 5, 6, 7].

the two orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ of type IIB, where a complete stringy calculation of the interaction among D-branes is possible, that the contribution of the massless open string states to the coefficient of the gauge kinetic term obtained from the annulus diagram, providing the inverse of the squared gauge coupling constant, is exactly equal, under open/closed string duality, to the contribution of the massless closed string states. Actually, in the cases that we have considered, it can also be shown that the contribution of the massive states is identically zero giving no threshold corrections. Hence the absence of threshold corrections makes open/closed string duality to work at the level of massless string states. This is the reason why one can use the SUGRA solutions to derive the perturbative behaviour of the dual gauge theory. In conclusion, it turns out that the validity of the gauge/gravity correspondence in these non-conformal and less supersymmetric theories that we have considered, is a direct consequence of the vanishing of threshold corrections. Actually, reading Ref. [18] where considerations similar to ours were made and applied to non-commutative gauge theories, we have become aware that the absence of threshold corrections in the case of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ was already noticed in Ref. [19] and was explained, following Ref. [20], as due to the fact that the open string massive states exchanged in the loop belong to supersymmetric long multiplets of $\mathcal{N}=2$ that are actually equal to short multiplets of $\mathcal{N}=4$. But, since short multiplets of $\mathcal{N}=4$ never contribute to the gauge coupling constant, one can conclude that in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ there cannot be any contribution from massive open string states circulating in the loop. The new point, as far as we know, is that these considerations can be directly extended to the case of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ that has fractional branes having $\mathcal{N}=1$ super Yang-Mills living on their world-volume, because the three twisted sectors of this orbifold are just three copies of the twisted sector of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. Therefore also in this case there are no threshold corrections.

In this review article we start by reminding what happens in the theories where the gauge/gravity correspondence beautifully works and then we try to see what happens in a certain number of non-supersymmetric theories. In these cases, however, we do not construct the "supergravity dual" solution corresponding to a system of D3 branes, in order to reproduce from it the perturbative properties of the gauge theory, but we write down instead the one-loop vacuum amplitude of an open string stretching between a D3 brane with a background gauge field turned-on on its world-volume and a stack of N D3 branes. This procedure is in fact closely related to the construction of the "supergravity dual" because, under open/closed string duality, one can also regard the previous amplitude as the interaction of the "dressed" brane and the stack of N D3 branes via the exchange of a closed string propagator and this in turn, in the low-energy limit, encodes the information about the large distance behaviour of the classical supergravity solution. [7, 21]

The first simple non-supersymmetric string theory we consider is the bosonic string theory in the orbifold $\mathbb{C}^{\delta/2}/\mathbb{Z}_2$, with $\delta \leq 22$. Explicit calculations show that, differently from the type IIB case, the threshold corrections to the gauge kinetic term do not vanish. Moreover the contribution of massless states in the closed channel turns out to be zero in

the field theory limit, showing that supergravity does not give any information about the gauge theory parameters. Therefore the gauge/gravity correspondence does not hold in this case. On the other hand, this theory is not consistent because of the presence of both open and closed string tachyons and therefore we do not discuss it any further.

Other natural candidates for non-supersymmetric theories are the type 0 ones that have been studied by constructing their supergravity duals in Ref.s [22, 23, 24]. Such theories exhibit, however, also the problem of having a tachyon in the closed string NS-NS sector. Moreover one finds that the zero-force condition among identical branes is not satisfied. This problem can be solved by considering a dyonic configuration of branes, made of N electric and N magnetic D3 branes. The gauge theory living on the world-volume of such a brane configuration is a $U(N)\times U(N)$ gauge theory with one gauge vector, six adjoint scalars for each gauge factor and four Weyl fermions in the bifundamental representation of the gauge group $(N, \bar{N})$ and $(\bar{N}, N)$. It exhibits a Bose-Fermi degeneracy at each mass level of the open string spectrum, which therefore guarantees the absence of interaction among the branes, at least at the lowest order in g_s. The interaction between a stack of N dyonic D3 branes and one extra dyonic D3 brane dressed with an $SU(N)$ gauge field turns out to be, in this theory, twice the corresponding one in type IIB. In particular, the coefficient of the gauge kinetic term is identically zero, yielding the right vanishing beta function both in the open and in the closed channel. Hence, for dyonic branes the gauge/gravity correspondence holds. When this theory is put in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ or $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, then the interaction between a stack of N dyonic fractional branes and an electric or magnetic fractional brane dressed with an $SU(N)$ gauge field turns out to be the same as in type IIB in that orbifold. Therefore, all the features discussed for that theory, as the validity of the gauge/gravity correspondence, are also shared by these non-supersymmetric models. The gauge theories living on the world-volume of such a brane configuration provide an example of the so-called *orbifold field theories*[25, 26, 27, 28] which are non-supersymmetric gauge theories that in the planar limit are perturbatively equivalent to some supersymmetric one.

Tachyon free orientifolds of type 0 theories, called $0'$ theories, were introduced in Ref. [29] and their properties were extensively studied from different points of view in Ref.s [30, 31, 32]. This is the theory we consider next in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. When one computes in type $0'$ the one-loop vacuum amplitude of an open string stretching between a stack of N fractional D3 branes and one brane of the same kind dressed with an external gauge field $SU(N)$, the result obtained in type IIB in the same orbifold is recovered. In particular, as in type IIB, the threshold corrections to the gauge theory parameters vanish: we find also in this case that this condition is crucial for the validity of the gauge/gravity correspondence.

Recently non-supersymmetric and non-conformal theories have been studied that in the large number of colours are equivalent to supersymmetric theories [2]. They are based

[2]See the recent review by Armoni, Shifman and Veneziano Ref. [34] and Ref.s therein. In Ref. [35] $1/N$ corrections are analysed.

on orientifolds of the 0B theory and go under the name of *orientifold field theories*. In Ref.s [31] and [33], non-supersymmetric gauge theories that are conformal in the planar limit have been discussed. One of them lives on the world-volume of N D3 branes of the orientifold $\Omega' I_6 (-1)^{F_L}$ of the 0B theory, where Ω' is the world-sheet parity [3], I_6 the inversion of the coordinates orthogonal to the world-volume of the D3 branes and F_L is the space-time fermion number operator in the left sector. This gauge theory is an example of orientifold field theory being, in the large N limit, equivalent to $\mathcal{N} = 4$ super Yang-Mills. It contains one gluon, six scalars in the adjoint representation and four Dirac fermions transforming according to the two-index (anti)symmetric representation of the gauge group $U(N)$ [4].

More recently some attention has been paid to the orientifold field theories that contain a gluon and a fermion transforming according to the two-index symmetric or antisymmetric representation of the gauge group $SU(N)$ [37] and that in the large N limit are equivalent to $\mathcal{N} = 1$ SYM.

In Ref. [38] the complete stringy description of the orientifold field theory whose spectrum has, in the large N limit, the same number of degrees of freedom as $\mathcal{N} = 2, 1$ super Yang-Mills, is provided by considering the orbifold projections $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ of the orientifold $0B/\Omega' I_6 (-1)^{F_L}$. In Ref. [34] the latter theory has been shown to be planar equivalent, both at perturbative and non-perturbative level, to $\mathcal{N} = 1$ SYM .

In Ref. [38] the running coupling constant has been computed in the open string framework and it has been shown that in the large N limit, where the Bose-Fermi degeneracy of the gauge theory is recovered and the threshold corrections vanish, one can obtain the perturbative behaviour of the orientifold field theories also from the closed string channel. However the next-to-leading term in the large N expansion of the β-function cannot be obtained from the closed string channel. This means that, as far as the running coupling constant is concerned, the gauge/gravity correspondence holds only in the planar limit. When considering the θ-angle instead, one can see that both the leading and the next-to-leading terms can be equivalently determined from the open and the closed string channel. This follows from the fact that in the string framework the θ-angle does not admit threshold corrections.

From the analysis of the above models we can conclude that the gauge/gravity correspondence is not a property concerning only supersymmetric theories, as type IIB. It may work as well in non-supersymmetric models and when the threshold corrections vanish (i.e. the contributions of massive states to the gauge theory parameters are zero) it admits a stringy description in terms of open/closed string duality. When this condition is satisfied, indeed, the contribution of the massless states in the open channel is mapped, under the modular transformation representing the open/closed string duality, into the corresponding one in the closed channel, allowing the gauge/gravity correspondence to

[3] We denote the world-sheet parity by Ω' because its action on the string states is not quite the same as the world-sheet parity Ω that is usually used for constructing type I ten-dimensional theory.

[4] The gravity dual of this theory has been constructed in Ref. [36].

hold.

The paper is organized as follows. In Sect. 2 we review the philosophy of the gauge/gravity correspondence, deriving general expressions for the holographic identifications valid both for fractional D3 branes and wrapped D5 branes. We also illustrate our procedure to get a stringy interpretation of this correspondence at the perturbative level in terms of open/closed string duality and its connection with the background field method. Sect. 3 is devoted to the analysis of the gauge/gravity correspondence in the supersymmetric cases of type IIB in the orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. We first use the holographic relations to derive the gauge theory parameters from the SUGRA solution and then we show that this correspondence is a direct consequence of open/closed string duality by evaluating the one-loop vacuum amplitude of an open string stretching between a stack of N D3 fractional branes and a further brane dressed with an $SU(N)$ background field. From Sect. 4 to Sect. 7 we apply the same procedure to non-supersymmetric string models. In Sect. 4 we discuss the case of the bosonic string in the orbifold $\mathbb{C}^{\delta/2}/\mathbb{Z}_2$ showing that the presence of threshold corrections and of the closed string tachyon do not allow the massless states in the closed string channel to reproduce the behaviour of the gauge-theory parameters. Sect. 5 is devoted to the case of type 0B string: we first review the structure of its open and closed string spectrum and that of the boundary state and then we explore the gauge/gravity correspondence in the case of dyonic branes configurations. In Sect. 6 we analyse the case of type $0'$ theories, discussing in some detail the structure of its open and closed string spectrum and the boundary state description of the branes. Then we show that, also in this case, the gauge/gravity correspondence holds and it follows from open/closed string duality. In Sect. 7 we discuss another orientifold of type 0B which is type $0\mathrm{B}/\Omega' I_6 (-1)^{F_L}$ and its orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. We analyse the open and closed string spectrum of these theories, their interpretation as orientifold field theories and then explore the gauge/gravity correspondence with the usual strategy. In Sect. 8 we summarize the main results and illustrate the conclusions of our work. Finally there are three appendices devoted respectively to the Θ-functions and their properties under modular transformations, to the explicit derivation of some results mentioned in various parts of the paper and to the Euler-Heisenberg actions that can be obtained for the various theories discussed in this article by performing the field theory limit.

2 The Philosophy of Gauge/Gravity Correspondence and its Stringy Interpretration

The gauge/gravity correspondence follows from the twofold nature of Dp branes which admit two alternative descriptions: a closed string description in which they appear as sources of closed strings that are emitted in the entire ten-dimensional space and an open string description in which they appear as hyperplanes where open strings are attached with their end-points satisfying appropriate boundary conditions. In the first perspective the massless closed string states emitted in the bulk generate non trivial SUGRA profiles,

while in the second one the open string massless fluctuations give rise to the existence of a $(p+1)$-dimensional gauge theory living on the brane world-volume. This twofold nature allows one to derive the gauge-theory quantum properties from the knowledge of the Dp brane classical geometry leading to the existence of some *holographic identifications* which relate the gauge theory parameters to the supergravity fields:

$$\frac{1}{g_{YM}^2} = f(\text{SUGRA fields}) \quad , \quad \theta_{YM} = g(\text{SUGRA fields}) \ , \tag{1}$$

where f and g are some particular functions. In particular, this correspondence relates the weak coupling regime of the gauge theory to the long distance behaviour of the SUGRA solution and the strong coupling regime to its near horizon limit. Indeed the typical structure of the SUGRA solution involves harmonic functions depending on the ratio $g_s N/r^{7-p}$ and then a large r expansion is formally equivalent to an expansion for small values of $g_s N$ (weak 't Hooft coupling) and viceversa. Therefore the amount of information that SUGRA can give about the gauge theory, by means of the holographic relations, depends on the specific case one is dealing with. Generally speaking, whenever the SUGRA solution is well-defined everywhere, holographic identifications should give (in principle) both perturbative and non-perturbative information about the gauge theory. This is what happens in the case of the Maldacena-Núñez solution [8] for which the SUGRA solution - which is not affected by any singularity - has been shown to encode the presence of a gaugino condensate [39] and has been used to derive the complete perturbative NSVZ β-function of the pure $\mathcal{N}=1$ SYM theory with gauge group $SU(N)$ [40] with, in addition, non-perturbative corrections due to fractional instantons. [41, 42, 43] These properties of $\mathcal{N}=1$ super Yang-Mills have also been derived from the regular Klebanov-Strassler solution [4, 44, 45, 46] that is also free of singularities. Instead in the cases of fractional branes in orbifolds the SUGRA solutions are affected by naked singularities and thus they cannot be trusted in the near horizon limit. Therefore it is not possible to use them in order to get non-perturbative information about the dual gauge theory: the holographic identification may be used only at the perturbative level (unless one considers specific deformations of the singular spaces as in Ref.s [47, 48]).

Let us briefly illustrate how to derive these gauge/gravity relations for the gauge theory living on fractional D3 and wrapped D5 branes using supergravity calculations. Since also the fractional D3 branes are D5 branes wrapped on a vanishing 2-cycle located at the orbifold fixed point, we can start from the *world-volume action* of a D5 brane, that is given by:

$$S = S_{BI} + S_{WZW} \ , \tag{2}$$

where the *Born-Infeld action* S_{BI} reads as:

$$S_{BI} = -\tau_5 \int d^6\xi e^{-\phi} \sqrt{-\det(G_{IJ} + B_{IJ} + 2\pi\alpha' F_{IJ})} \ , \ \tau_5 = \frac{1}{g_s\sqrt{\alpha'}(2\pi\sqrt{\alpha'})^5} \ , \tag{3}$$

while the *Wess-Zumino-Witten action* S_{WZW} is given by:

$$S_{WZW} = \tau_5 \int_{V_6} \left[\sum_n C_n \wedge e^{2\pi\alpha' F + B_2} \right] . \tag{4}$$

We divide the six-dimensional world-volume into four flat directions in which the gauge theory lives and two directions on which the brane is wrapped. Let us denote them with the indices $I, J = (\alpha, \beta; A, B)$ where α and β denote the flat four-dimensional ones and A and B the wrapped ones. As usual, we assume the supergravity fields to be independent from the coordinates α, β. We also assume that the determinant in Eq. (3) factorizes into a product of two determinants, one corresponding to the four-dimensional flat directions where the gauge theory lives and the other one corresponding to the wrapped ones where we have only the metric and the NS-NS two-form field. By expanding the first determinant and keeping only the quadratic term in the gauge field we obtain:

$$(S_{BI})_2 = -\tau_5 \frac{(2\pi\alpha')^2}{8} \int d^6\xi e^{-\phi} \sqrt{-\det G_{\alpha\beta}} G^{\alpha\gamma} G^{\beta\delta} F^a_{\alpha\beta} F^a_{\gamma\delta} \sqrt{\det\left(G_{AB} + B_{AB}\right)}, \qquad (5)$$

where we have included a factor 1/2 coming from the normalization of the gauge group generators $\mathrm{Tr}[T^a T^b] = \frac{\delta^{ab}}{2}$.

We assume that along the flat four-dimensional directions the metric is the Minkowski one apart from the warp factor, while along the wrapped ones, in addition to the warp factor, there is also a non-trivial metric. This means that the longitudinal part of the metric can be written as

$$ds^2 = H^{-1/2}\left(dx^2_{3,1} + ds^2_2\right). \qquad (6)$$

By inserting this metric in Eq. (5) we see that the warp factor cancels out in the Yang-Mills action and from it we can then extract the inverse of the squared gauge coupling constant as the coefficient of the gauge kinetic term $-\frac{1}{4}\int d^4x F^{a\,\alpha\beta} F^a_{\alpha\beta}$:

$$\frac{4\pi}{g^2_{YM}} = \frac{1}{g_s (2\pi\sqrt{\alpha'})^2} \int_{\mathcal{C}_2} d^2\xi e^{-\phi} \sqrt{\det\left(G_{AB} + B_{AB}\right)} \quad . \qquad (7)$$

This formula is valid for both wrapped and fractional branes of the orbifolds having only one vanishing 2-cycle as the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. $\mathcal{C}_2$ is the cycle around which the branes are wrapped. The θ angle, in the case of both fractional D3 branes and wrapped D5 branes, can be obtained by extracting the coefficient of the term $\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta} \tilde{F}^{a\,\alpha\beta}$ from S_{WZW} getting:

$$\theta_{YM} = \tau_5 (2\pi\alpha')^2 (2\pi)^2 \int_{\mathcal{C}_2} (C_2 + C_0 B_2) = \frac{1}{2\pi\alpha' g_s} \int_{\mathcal{C}_2} (C_2 + C_0 B_2) \quad . \qquad (8)$$

Eq.s (7) and (8) provide an explicit realization of the holographic identifications (1), establishing a relation between quantities peculiar of the gauge theory living on the world-volume of the D3 branes and the supergravity

fields. We want to stress that these relations are not based on the probe analysis; they have a more general validity as stressed in Ref. [12] and are therefore also valid in the case of supersymmetric $\mathcal{N} = 1$ theories where the probe analysis cannot be done. Before proceeding further, it is interesting to notice that in the case of fractional branes Eq.s (7) and (8) can be written in a single expression:

$$\tau_{YM} \equiv \frac{\theta_{YM}}{2\pi} + i\frac{4\pi}{g^2_{YM}} = \frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \int_{\mathcal{C}_2} (C_2 + \tau B_2) \quad , \quad \tau = C_0 + ie^{-\phi}. \qquad (9)$$

After defining the quantity $G_3 \equiv d(C_2 + \tau B_2)$, Eq. (9) can be rewritten in the following form:

$$\tau_{YM} = \frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \int_{\mathcal{B}} G_3 \quad , \tag{10}$$

where $\mathcal{B}$ is the 3-cycle given by the direct product of the original 2-cycle $\mathcal{C}_2$ with a suitable non-compact 1-cycle living in the plane orthogonal to both the branes and the orbifold. Eq.s (9) and (10) can also be extended to the case of fractional branes in the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ where the gauge theory living on the world-volume of the D3 branes preserves four supersymmetry charges. In this case we have [62, 70]

$$\tau_{YM} = \frac{1}{2(2\pi\sqrt{\alpha'})^2 g_s} \sum_{i=1}^{3} \int_{\mathcal{C}_2^i} (C_2 + \tau B_2) \quad , \tag{11}$$

being $\mathcal{C}_2^i$ the exceptional shrinking 2-cycle of the orbifold geometry. By defining the non-compact 3-cycle $\mathcal{B}$

$$\mathcal{B} \equiv \bigcup_{\ell=1}^{3} \mathcal{B}_\ell \qquad \text{with} \qquad \mathcal{B}_\ell \equiv \bigcup_{i=1}^{3} \mathcal{C}_2^i \times \beta_\ell \quad , \tag{12}$$

where β_ℓ is a suitable non-compact 1-cycle living in the plane $z^\ell = x^{2\ell+2} + ix^{2\ell+3}$ ($\ell = 1, 2, 3$) orthogonal to the brane, we can write

$$\tau_{YM} = \frac{1}{2(2\pi\sqrt{\alpha'})^2 g_s} \int_{\mathcal{B}} G_3 \quad . \tag{13}$$

This provides a generalization to the case of a non constant axion and dilaton of the formulas used in computing the parameters of the gauge theory after the geometric transition. [47]

The aim of this review is to discuss the stringy interpretation of the gauge/gravity correspondence. In particular in Ref. [17] we have elaborated a strategy which has allowed us to understand why the SUGRA solution is able to reproduce the gauge theory parameters, at the first order of their perturbative expansion. Let us review the main features of this procedure.

We first remind that in field theory the one-loop running coupling constant may be determined through the background field method, by calculating the one-loop correction to the two-point function involving two external background field strengths F (see Fig. 1) and reading its contribution to the gauge kinetic term. In the context of string theory, the open string amplitude reducing to the previous one in the field theory limit is given by the one-loop vacuum amplitude of an open string stretching between a stack of N D3 branes and a further D3 brane with a background $SU(N)$ gauge field turned-on on its world-volume. From it we can extract the second order term in the background field, selecting the amplitude shown in Fig. 2, which gives the full open string one-loop correction to the two-point function with two external background field strengths F. This means that, as

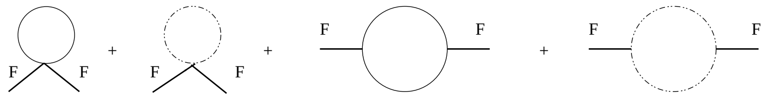

Figure 1: One-loop correction to the two-point function with two external background field strengths. These are the diagrams one has to consider when evaluating the one loop running coupling constant of the gauge theory via the background field method. Dashed lines denote the ghost fields.

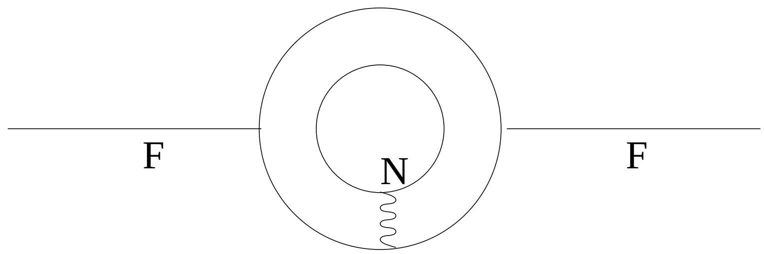

Figure 2: Second order expansion in the background field of the one-loop vacuum amplitude of an open string stretching between a stack of N D3 branes and a further brane dressed with an $SU(N)$ background field. The number N of D3 branes encodes the information about the non-abelian nature of the quantum gauge fields propagating in the loop. In the field theory limit this diagram reduces to those in Fig. 1

in field theory, one can read the gauge theory parameters with all the string corrections as follows:

$$\frac{1}{g_{YM}^2} \quad \text{as the coefficient of the gauge kinetic term} \quad -\frac{1}{4}\int d^4x F^{a\,\alpha\beta}F^a_{\alpha\beta}, \tag{14}$$

$$\theta_{YM} \quad \text{as the coefficient of the topological charge term} \quad \frac{1}{32\pi^2}\int d^4x F^{a\,\alpha\beta}\tilde{F}^a_{\alpha\beta}. \tag{15}$$

Obviously, by performing the field theory limit in the open string channel, only massless open string states (i.e. the gauge degrees of freedom) propagate in the loop, and therefore this procedure to evaluate the gauge theory parameters coincides with the one of the background field method in field theory.

One can also rewrite the one-loop vacuum amplitude in the closed string channel and identify again the running coupling constant and the θ-angle, at the full closed string level, respectively as the coefficient of the gauge kinetic term and of the topological charge term in the F expansion, obtaining for those quantities the same result as in the open string channel. This is a direct consequence of the open/closed string duality. What is not obvious and actually in general not true, however, is that the contribution of the massless open string states circulating in the loop is equal to that of the massless closed string states exchanged between the branes and selected by performing the large distance limit between the branes .[7, 21] We show that this is exactly what happens in those cases in which the supergravity solution is able to reproduce the gauge theory parameters. Hence, this means that if the contribution of the open massless string states is mapped, under open/closed

string duality, into the contribution of closed string massless states, then the supergravity solution contains the full information about the gauge theory parameters. Notice that the above technique gives a quantitative explanation of why the SUGRA fields reproduce the β-function and θ-angle at one loop.

In this section we have computed the gauge parameters in two apriori different ways obtaining respectively Eq.s (7), (8) and Eq.s (14), (15). In the first case we have used the Born-Infeld action with the inclusion of the WZW term and we have determined the gauge parameters in terms of the supergravity fields given by a classical solution of the SUGRA equations of motion describing N fractional D3 or wrapped D5 branes, while in the second case we have performed a complete stringy calculation, that can be done only in the case of fractional branes, computing the interaction between a fractional D3 brane having a non-abelian gauge field on its world-volume and a system of N fractional D3 branes. In both cases no use has been done of the probe technique and in general we expect to obtain two different results because the stringy calculation includes also the contribution of the massive string states and in general the contribution of the massless open string states circulating in the loop is not mapped under open/closed string duality into that of the massless closed string states that are the only ones appearing in the approach based on supergravity. We will see, however, that in many interesting cases the massless states appearing in the two channels are precisely mapped into each other and in this case the supergravity approach provides the correct perturbative behaviour of the gauge theory living on the world-volume of N D3 branes.

One could in principle generalize it to the full perturbative level by considering multi-loop open string amplitudes and converting them into the corresponding multiboundaries tree level amplitudes in the closed channel, but calculations would be of course much more involved.

3 Gauge/Gravity Correspondence in Supersymmetric String Theory

In this section we first use fractional branes of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ to show that the perturbative behaviour of the gauge theory living on their world-volume, namely $\mathcal{N}=2$ super Yang-Mills, can be reproduced from their corresponding classical solution through the gauge/gravity relations. Then we show that, working in the pure string framework and using only open/closed string duality, the perturbative properties of the gauge theory living on the fractional D-branes can be derived not only in the open string channel, as expected, but also in the closed string channel. We finally show that this at first sight surprising result turns out to be, in these models, a direct consequence of the absence of threshold corrections to the running gauge coupling constant.

3.1 Gauge/gravity correspondence in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$

In this subsection we consider fractional D3 and D7 branes of the non-compact orbifold $\mathbb{C}^2/\mathbb{Z}_2$ in order to study the properties of $\mathcal{N}=2$ super QCD. We group the coordinates of the directions $(x^4,\ldots,x^9)$ transverse to the world-volume of the D3 brane where the gauge theory lives, into three complex quantities:

$$z_1 = x^4 + ix^5 \quad , \quad z_2 = x^6 + ix^7 \quad , \quad z_3 = x^8 + ix^9. \tag{16}$$

The non trivial generator h of Z_2 acts as

$$z_2 \to -z_2 \quad , \quad z_3 \to -z_3 \tag{17}$$

leaving z_1 invariant, showing the presence of one fixed point at the origin corresponding to a vanishing 2-cycle located at $z_2 = z_3 = 0$. Fractional D3 branes are D5 branes wrapped on the vanishing 2-cycle and therefore are, unlike bulk branes, stuck at the orbifold fixed point. By considering N fractional D3 and M fractional D7 branes of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ we are able to study $\mathcal{N}=2$ super QCD with M hypermultiplets. In order to do that, we need to determine the classical solution corresponding to the previous brane configuration. For the case of the orbifold $\mathbb{C}^2/Z_2$ the complete classical solution was found in Ref. [49] [5]. In the following we write it explicitly for a system of N fractional D3 branes with their world-volume along the directions x^0, x^1, x^2, and x^3 and M fractional D7 branes containing the D3 branes in their world-volume and having the remaining four world-volume directions along the orbifolded ones. The metric, the 5-form field strength, the axion and the dilaton are given by [6]:

$$\begin{aligned} ds^2 &= H^{-1/2}\,\eta_{\alpha\beta}\,dx^\alpha dx^\beta + H^{1/2}\left(\delta_{\ell m}\,dx^\ell dx^m + \mathrm{e}^{-\phi}\delta_{ij}dx^i dx^j\right) \quad , & (18)\\ \widetilde{F}_{(5)} &= d\left(H^{-1}\,dx^0\wedge\cdots\wedge dx^3\right) + {}^*d\left(H^{-1}\,dx^0\wedge\cdots\wedge dx^3\right) \quad , & (19) \end{aligned}$$

$$\tau \equiv C_0 + i\mathrm{e}^{-\phi} = \mathrm{i}\left(1 - \frac{Mg_s}{2\pi}\log\frac{z}{\epsilon}\right) \quad , \quad z \equiv x^4 + ix^5 = y\mathrm{e}^{i\theta} \quad , \tag{20}$$

where the self-dual field strength $\tilde{F}_{(5)}$ is given in terms of the NS-NS and R-R 2-forms B_2 and C_2 and of the 4-form potential C_4 by $\tilde{F}_{(5)} = dC_4 + C_2\wedge dB_2$. The warp factor H is a function of the coordinates $(x^4,\ldots,x^9)$ and ϵ is an infrared cutoff. The twisted fields are instead given by $B_2 = \omega_2 b$, $C_2 = \omega_2 c$ where ω_2 is the volume form of the vanishing 2-cycle and

$$b\mathrm{e}^{-\phi} = \frac{(2\pi\sqrt{\alpha'})^2}{2}\left[1 + \frac{2N-M}{\pi}g_s\log\frac{y}{\epsilon}\right] \quad , \quad c + C_0 b = -2\pi\alpha'\theta g_s(2N-M) \quad . \tag{21}$$

It can be seen that the previous solution has a naked singularity of the repulson type at short distances. But, on the other hand, if we use a brane probe approaching from

[5] See also Ref.s [50, 51, 52, 53, 54] and Ref. [3] for a review on fractional branes.

[6] We denote by α and β the four directions corresponding to the world-volume of the fractional D3 brane, by ℓ and m those along the four orbifolded directions x^6, x^7, x^8 and x^9 and by i and j the directions x^4 and x^5 that are transverse to both the D3 and the D7 branes.

infinity the stack of branes, described by the previous classical solution, it can also be seen that the tension of the probe vanishes at a distance that is larger than that of the naked singularity. The point where the probe brane becomes tensionless is called in the literature *enhançon* [55] and at this point the classical solution does not describe anymore the stack of fractional branes.

Let us now exploit the gauge/gravity relations derived in the previous section, to determine the coupling constants of the world-volume theory from the supergravity solution. In the case of fractional D3 branes of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, that is characterized by one single vanishing 2-cycle $\mathcal{C}_2$, the gauge coupling constant given in Eq. (7) reduces to

$$\frac{1}{g_{YM}^2} = \frac{\tau_5 (2\pi\alpha')^2}{2} \int_{\mathcal{C}_2} e^{-\phi} B_2 = \frac{1}{4\pi g_s (2\pi\sqrt{\alpha'})^2} \int_{\mathcal{C}_2} e^{-\phi} B_2 \quad . \tag{22}$$

By inserting in Eq.s (22) and (8) the classical solution we get the following expressions for the gauge coupling constant and the θ_{YM} angle: [49]

$$\frac{1}{g_{YM}^2} = \frac{1}{8\pi g_s} + \frac{2N-M}{16\pi^2} \log \frac{y^2}{\epsilon^2} \quad , \quad \theta_{YM} = -\theta(2N-M) \quad . \tag{23}$$

Notice that the gauge coupling constant appearing in the previous equation is the <u>bare</u> gauge coupling constant computed at the scale $m \sim y/\alpha'$, while the square of the bare gauge coupling constant computed at the cut-off $\Lambda \sim \epsilon/\alpha'$ is equal to $8\pi g_s$.

In the case of an $\mathcal{N}=2$ supersymmetric gauge theory the gauge multiplet contains a complex scalar field Ψ whose action term can be found when deriving the Yang-Mills action from the Born-Infeld one. In fact the derivation of the kinetic term for the scalar field is obtained from the term in the Born-Infeld action depending on the brane coordinates x^4 and x^5 that are transverse to both the branes and the orbifold. This implies that the complex scalar field of the gauge supermultiplet is related to the coordinate z of supergravity through the following gauge/gravity relation $\Psi \sim \frac{z}{2\pi\alpha'}$. This is another example of holographic identification between a quantity, Ψ, peculiar of the gauge theory living on the fractional D3 branes and another one, the coordinate z, peculiar of supergravity. It allows one to obtain the gauge theory anomalies from the supergravity background. In fact, since we know how the scale and $U(1)$ transformations act on Ψ, from the previous gauge/gravity relation we can deduce how they act on z, namely

$$\Psi \rightarrow s e^{2i\alpha} \Psi \Longleftrightarrow z \rightarrow s e^{2i\alpha} z \Longrightarrow y \rightarrow sy \quad , \quad \theta \rightarrow \theta + 2\alpha \quad . \tag{24}$$

Those transformations do not leave invariant the supergravity background in Eq. (21) and when we use them in Eq.s (22) and (8), they generate the anomalies of the gauge theory living on the fractional D3 branes. In fact, by acting with those transformations in Eq.s (23), we get:

$$\frac{1}{g_{YM}^2} \rightarrow \frac{1}{g_{YM}^2} + \frac{2N-M}{8\pi^2} \log s \quad , \quad \theta_{YM} \rightarrow \theta_{YM} - 2\alpha(2N-M) \quad . \tag{25}$$

The first equation generates the β-function of $\mathcal{N} = 2$ super QCD with M hypermultiplets given by:

$$\beta(g_{YM}) = -\frac{2N - M}{16\pi^2} g_{YM}^3 \quad , \tag{26}$$

while the second one reproduces the chiral $U(1)$ anomaly. [11, 12] In particular, if we choose $\alpha = \frac{2\pi}{2(2N-M)}$, then θ_{YM} is shifted by a factor 2π. But since θ_{YM} is periodic of 2π, this means that the subgroup $Z_{2(2N-M)}$ is not anomalous in perfect agreement with the gauge theory results.

From Eq.s (23) it is easy to compute the combination:

$$\tau_{YM} \equiv \frac{\theta_{YM}}{2\pi} + i\frac{4\pi}{g_{YM}^2} = i\frac{2N - M}{2\pi} \log \frac{z}{y_e} \quad , \quad y_e = \epsilon \mathrm{e}^{-\pi/[(2N-M)g_s]} \quad , \tag{27}$$

where y_e is the enhançon radius and corresponds, in the gauge theory, to the dimensional scale generated by dimensional transmutation that we call Λ_{QCD} in order not to confuse it with the cut-off Λ. Eq. (27) reproduces the perturbative moduli space of $\mathcal{N} = 2$ super QCD, but not the instanton corrections. This is consistent with the fact that the classical solution is reliable for large distances in supergravity corresponding to short distances in the gauge theory, while it cannot be used below the enhançon radius where non-perturbative physics is expected to show up. Notice that the quantity G_3, defined in the previous section, results to be:

$$G_3 \equiv d(C_2 + \tau B_2) = i(2\pi\sqrt{\alpha'})^2 g_s \cdot \frac{2N - M}{2\pi} \omega_2 \wedge \frac{dz}{z} \tag{28}$$

and we get the following identities

$$\frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \int_{\mathcal{B}} G_3 = \tau_{YM} \quad , \quad \frac{1}{(2\pi\sqrt{\alpha'})^2 g_s} \int_{\mathcal{A}} G_3 = N - \frac{M}{2} \quad . \tag{29}$$

Hence Eq.s (27) and (28) provide an explicit realization of the holographic identity given in Eq. (10). $\mathcal{B}$ is the non-compact 3-cycle consisting of $\mathcal{C}_2$ times a 1-cycle along which we have to integrate between the IR cutoff y_e and an UV one $z \equiv ye^{i\theta}$. $\mathcal{A}$ is a compact 3-cycle consisting of $\mathcal{C}_2$ and a 1-cycle along which we have to integrate between $-y_e$ and y_e. Notice, however, that what we call UV cutoff according to the notation followed in the literature, should be more properly called the scale $m \sim y/\alpha'$ at which we compute the <u>bare</u> gauge coupling constant, while the UV cutoff of the gauge theory is actually $\Lambda \sim \epsilon/\alpha'$.

Eq.s (29) are precisely the ones used in the approach followed in Ref. [47] for $M = 0$.

The previous results can also be extended to the case of fractional D3/D7 branes of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In this case only the large distance behaviour of the classical solution is known.[62] In particular the solution for the twisted fields is:

$$G_3 = \frac{i}{\pi}(2\pi\sqrt{\alpha'})^2 g_s \cdot \sum_{i=1}^{3} \left[\omega_2^i \wedge \left(N\frac{dz_i}{z_i} - \frac{M}{6}\frac{dz_1}{z_1} - \frac{M}{2}\delta_{i1}\frac{dz_1}{z_1} \right) \right] , \tag{30}$$

where N is the number of fractional D3 branes of type 1 and M is the number of D7 branes of type 3 and 4 in the notation of Ref. [62] and the following expression for the background value of the B_2 field has been used:

$$\int_{\mathcal{C}_2^i} B_2 = \frac{4\pi^2\alpha'}{6} \quad . \tag{31}$$

By integrating the three form in Eq.(30) on the non-compact $\mathcal{B}$ cycle defined in Eq. (12) we get:

$$\tau_{YM} = \frac{1}{2(2\pi\sqrt{\alpha'})^2 g_s}\int_{\mathcal{B}} G_3 = \frac{i}{2\pi}(3N-M)\log(z/y_e) \quad . \tag{32}$$

that is the correct one-loop expression of the gauge coupling constant of $\mathcal{N}=1$ super QCD. Moreover one has:

$$\frac{1}{2(2\pi\sqrt{\alpha'})^2 g_s}\int_{\mathcal{A}_\ell} G_3 = N - M\delta_{\ell 1} \quad , \tag{33}$$

where A_ℓ is the compact 3-cycle consisting of $\bigcup_{i=1}^3 C_2^i$ and of a 1-cycle along the direction ℓ which we have to integrate between $-y_e$ and y_e, being $y_e = \epsilon e^{-\pi/[2(3N-M)g_s]}$. The last two equations give a generalization of the relations obtained in Ref. [47] to a case with running dilaton and axion.

In conclusion, we have derived the perturbative behaviour of $\mathcal{N}=2$ super QCD with M flavours by using the holographic identifications, given in Eq.s (22) and (8). This apriori unexpected result will be clarified in the next subsection by computing the annulus diagram in the full string theory and showing that the threshold corrections to the gauge coupling constant vanish. This means that, under open/closed string duality, the contribution of the massless open string states is precisely mapped into that of the massless closed string states and that therefore the supergravity solution is sufficient to derive the perturbative behaviour of the gauge theory.

3.2 Gauge/gravity correspondence from open/closed duality: $\mathcal{N}=2$ case

In this section we compute the one-loop vacuum amplitude of an open string stretching between a fractional D3 brane of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ *dressed* with a background $SU(N)$ gauge field on its world-volume and a stack of N ordinary fractional D3 branes. This can be equivalently done by computing the tree closed string diagram containing two boundary states and a closed string propagator.

We are interested in the case of parallel fractional D3 branes with their world-volume along the directions x^0, x^1, x^2, x^3, that are completely external to the space on which the orbifold acts. The background gauge field lives on the four-dimensional world-volume of the fractional D3 brane and, without loss of generality, it can be taken to have the

following form:

$$\hat{F}_{\alpha\beta} \equiv 2\pi\alpha' F_{\alpha\beta} = \begin{pmatrix} 0 & f & 0 & 0 \\ -f & 0 & 0 & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & -g & 0 \end{pmatrix} \quad . \tag{34}$$

The free energy of an open string stretched between a dressed D3 brane and a stack of N D3 branes located at a distance y in the plane (x^4, x^5) that is orthogonal to both the world-volume of the D3 branes and the four-dimensional space on which the orbifold acts, is given by:

$$Z = N \int_0^\infty \frac{d\tau}{\tau} Tr_{NS-R} \left[\left(\frac{e+h}{2} \right) (-1)^{F_s} (-1)^{G_{bc}} P_{GSO} e^{-2\pi\tau L_0} \right] \equiv \quad Z_e^o + Z_h^o \tag{35}$$

where F_s is the space-time fermion number, G_{bc} is the ghost number and the GSO projector is given by:

$$P_{GSO} = \frac{(-1)^{G_{\beta\gamma}} + (-1)^F}{2} \quad , \tag{36}$$

with $G_{\beta\gamma}$ being the superghost number:

$$G_{\beta\gamma} = -\sum_{m=1/2}^{\infty} (\gamma_{-m}\beta_m + \beta_{-m}\gamma_m) \ , \ G_{\beta\gamma} = -\gamma_0\beta_0 - \sum_{m=1}^{\infty} (\gamma_{-m}\beta_m + \beta_{-m}\gamma_m) \tag{37}$$

respectively in the NS and in the R sector. F is the world-sheet fermion number defined by

$$F = \sum_{t=1/2}^{\infty} \psi_{-t} \cdot \psi_t - 1 \tag{38}$$

in the NS sector and by

$$(-1)^F = \Gamma^{11} (-1)^{F_R} \quad , \quad \Gamma^{11} \equiv \Gamma^0 \Gamma^1 \ldots \Gamma^9 \quad , \quad F_R = \sum_{n=1}^{\infty} \psi_{-n} \cdot \psi_n \tag{39}$$

in the R sector.

The superscript o stands for open because we are computing the annulus diagram in the open string channel. The fact that we are considering a string theory in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ is encoded in the presence of the orbifold projector $P = (e+h)/2$ in the trace. The explicit computation can be found in Appendix B of Ref. [17]. Here we give only the final results:

$$\begin{aligned} Z_e^o &= -\frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \\ &\times \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{\sin\pi\nu_f \sin\pi\nu_g}{f_1^4(e^{-\pi\tau}) \Theta_1(i\nu_f\tau|i\tau) \Theta_1(i\nu_g\tau|i\tau)} \\ &\times \left[f_3^4(e^{-\pi\tau}) \Theta_3(i\nu_f\tau|i\tau) \Theta_3(i\nu_g\tau|i\tau) - f_4^4(e^{-\pi\tau}) \Theta_4(i\nu_f\tau|i\tau) \Theta_4(i\nu_g\tau|i\tau) \right. \\ &- \left. f_2^4(e^{-\pi\tau}) \Theta_2(i\nu_f\tau|i\tau) \Theta_2(i\nu_g\tau|i\tau) \right] \end{aligned} \tag{40}$$

and[7]

$$
\begin{aligned}
Z_h^o &= -\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})} \\
&\times \int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\right] \\
&\times \left[\Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)-\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau)\right] \\
&- \frac{iN}{32\pi^2}\int d^4x\,F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \quad ,
\end{aligned}
\tag{41}
$$

where $\tilde{F}_{\alpha\beta}=\frac{1}{2}\epsilon_{\alpha\beta\delta\gamma}F^{\delta\gamma}$. The Θ-functions are listed in A. In the previous equations we have defined:

$$
\nu_f\equiv\frac{1}{2\pi i}\log\frac{1+2\pi\alpha'\hat{f}}{1-2\pi\alpha'\hat{f}}\qquad\text{and}\qquad\nu_g\equiv\frac{1}{2\pi i}\log\frac{1-i2\pi\alpha'\hat{g}}{1+i2\pi\alpha'\hat{g}}\quad . \tag{42}
$$

The calculation of Z_e for the untwisted sector was originally done in Ref. [56] for the case of a D9 brane. The three terms in Eq. (40) come respectively from the NS, $NS(-1)^F$ and R sectors, while the contribution from the $R(-1)^F$ sector vanishes. In Eq. (41) the three terms come respectively from the $NS, NS(-1)^F$ and $R(-1)^F$ sectors, while the R contribution vanishes because the projector h annihilates the Ramond vacuum.

The above computation can also be performed in the *closed string channel* where Z_e^c and Z_h^c are now given by the tree level closed string amplitude between two untwisted and two twisted boundary states respectively:

$$
Z_e^c=\frac{\alpha'\pi N}{2}\int_0^\infty dt\ {}^U\langle D3;F|e^{-\pi t(L_0+\bar{L}_0)}|D3\rangle^U \tag{43}
$$

and

$$
Z_h^c=\frac{\alpha'\pi N}{2}\int_0^\infty dt\ {}^T\langle D3;F|e^{-\pi t(L_0+\bar{L}_0)}|D3\rangle^T \quad , \tag{44}
$$

where $|D3;F>$ is the boundary state dressed with the gauge field F. The details of this calculation are presented in Appendix C of Ref. [17]. Here we give only the final results that are

$$
\begin{aligned}
Z_e^c &= \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{dt}{t^3}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{\sin\pi\nu_f\sin\pi\nu_g}{\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)f_1^4(e^{-\pi t})} \\
&\quad\times\left\{f_3^4(e^{-\pi t})\Theta_3(\nu_f|it)\Theta_3(\nu_g|it)-f_2^4(e^{-\pi t})\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)\right. \\
&\quad\left.-f_4^4(e^{-\pi t})\Theta_4(\nu_f|it)\Theta_4(\nu_g|it)\right\}
\end{aligned}
\tag{45}
$$

and

$$
\begin{aligned}
Z_h^c &= \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_4^2(0|it)\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)} \\
&\quad\times\left\{\Theta_2^2(0|it)\Theta_3(\nu_f|it)\Theta_3(\nu_g|it)-\Theta_3^2(0|it)\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)\right\} \\
&\quad-\frac{iN}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\int\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} \quad .
\end{aligned}
\tag{46}
$$

[7]Notice that the sign of the topological term is opposite to the one appearing in the Ref.s [38, 17] because of the different definition of the operator N_0 given in Eq. (63) with respect to the corresponding operator given in Eq. (113) of Ref. [17].

The three terms in Eq. (45) respectively come from the NS-NS, R-R and NS-$NS(-1)^F$ sectors, while those in Eq. (46) from the NS-NS, R-R and R-$R(-1)^F$ sectors. In particular, the twisted odd R-$R(-1)^F$ spin structure gets a nonvanishing contribution only from the zero modes, as explicitly shown in Ref. [17].

It goes without saying that the two expressions for Z separately obtained in the open and the closed string channels are, as expected, equal to each other. This equality goes under the name of open/closed string duality and can be easily shown by using how the Θ functions transform (see Eq. (359)) under the modular transformation that relates the modular parameters in the open and closed string channels, namely $\tau = \frac{1}{t}$. It can be easily seen that, in going from the open (closed) to the closed (open) string channel, we have the following correspondence between the various non vanishing spin structures: [57]

$$\begin{aligned} &NS \leftrightarrow NS - NS \ , && NS(-1)^F \leftrightarrow R - R \\ &R \leftrightarrow NS - NS(-1)^F \ , && R(-1)^F \leftrightarrow R - R(-1)^F \ . \end{aligned} \tag{47}$$

The distance y between the dressed D3 brane and the stack of the N D3 branes makes the integral in Eq. (46) convergent for small values of t, while in the limit $t \to \infty$, the integral is logarithmically divergent. This divergence is due to a twisted tadpole corresponding to the exchange of massless closed string states between the two boundary states in Eq. (44). We would like here to stress that the presence of the gauge field makes the divergence to appear already *at the string level* before any field theory limit ($\alpha' \to 0$) is performed. When F vanishes, the divergence is eliminated by the integrand being identically zero as a consequence of the fact that the fractional branes are BPS states.

As observed in Ref.s [58, 59, 60, 61, 62] tadpole divergences correspond in general to the presence of gauge anomalies that make the gauge theory inconsistent and have to be eliminated by drastically modifying the theory or by fixing particular values of the parameters. For instance in type I superstring they are eliminated by fixing the gauge group to be $SO(32)$. As stressed in Ref.s [59, 60, 61, 62] logarithmic tadpole divergences do not instead correspond to gauge anomalies. In the bosonic string they have been cured in different ways. [63, 64, 65, 66, 67, 68] It turns out, in our case, that the logarithmic divergent tadpoles correspond to the fact that the gauge theory living on the brane is not conformal invariant and in fact they provide the correct one-loop running coupling constant. Following the suggestion of Ref.s [59, 60], we cure these divergences just by introducing in Eq. (46) an infrared cutoff that regularizes the contribution of the massless closed string states. Since in the open/closed string duality an infrared divergence in the closed string channel corresponds to an ultraviolet divergence in the open string channel it is easy to see that the expression in Eq. (41) is divergent for small values of τ and needs an ultraviolet cutoff. It will turn out that this divergence is exactly the one-loop divergence that one gets in $\mathcal{N} = 2$ super Yang-Mills that is the gauge theory living on the world-volume of the fractional D3 brane. Our results are also consistent with the approach of Ref. [69].

In the following we want to use the previous string calculation for deriving the co-

efficient of the quadratic term involving the gauge field and to show that only massless states give a non-vanishing contribution to it. The contribution of the massive states is identically zero and this implies the absence of threshold corrections.

To this aim it is useful to write Eq. (41) in a more convenient way. Using the notation for the Θ-functions given in Eq. (368) and the identity in Eq. (372) with

$$\begin{cases} h_i = g_1 = g_2 = 0 \\ g_3 = -g_4 = 1 \\ \nu_1 = i\nu_f\tau \ ; \ \nu_2 = i\nu_g\tau \ ; \ \nu_3 = \nu_4 = 0 \\ \nu'_1 = -\nu'_2 = \frac{i}{2}\left(\nu_g - \nu_f\right)\tau \ ; \ \nu'_3 = \nu'_4 = \frac{i}{2}\left(\nu_g + \nu_f\right)\tau \end{cases}$$

we can rewrite Eq. (41) as follows:

$$\begin{aligned} Z_h^o &= \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \\ &\times \left[\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\right]\left[\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)\right. \\ &- \left. 2\Theta_1\left(i\frac{\nu_g-\nu_f}{2}\tau|i\tau\right)\Theta_1\left(i\frac{\nu_f-\nu_g}{2}\tau|i\tau\right)\Theta_2^2\left(i\frac{\nu_f+\nu_g}{2}\tau|i\tau\right)\right] \\ &- \frac{iN}{32\pi^2}\int d^4x\, F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \quad , \end{aligned} \tag{48}$$

which turns out to be equal to

$$\begin{aligned} Z_h^o = &-\frac{2N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\right] \\ &\times\Theta_1\left(i\frac{\nu_g-\nu_f}{2}\tau|i\tau\right)\Theta_1\left(i\frac{\nu_f-\nu_g}{2}\tau|i\tau\right)\Theta_2^2\left(i\frac{\nu_f+\nu_g}{2}\tau|i\tau\right) . \end{aligned} \tag{49}$$

because the first and the third terms in Eq. (48) cancel each other. By expanding the previous equation up to the second order in F and using Eq. (367) together with $\nu_f \simeq -i\frac{f}{\pi}$ and $\nu_g \simeq -\frac{g}{\pi}$, we get:

$$Z_h^o = \frac{N}{32\pi^2}\int d^4x(F^a_{\alpha\beta}F^{a\,\alpha\beta} - iF^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta})\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}, \tag{50}$$

which reduces to

$$\begin{aligned} Z_h^o(F) \rightarrow &\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \frac{N}{8\pi^2}\int_{\frac{1}{\alpha'\Lambda^2}}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\right\} \\ &-iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{\frac{1}{\alpha'\Lambda^2}}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \quad . \end{aligned} \tag{51}$$

In the closed string channel we get instead:

$$\begin{aligned} Z_h^c(F) \rightarrow &\left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \frac{N}{8\pi^2}\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\right\} \\ &-iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} \quad . \end{aligned} \tag{52}$$

Notice that in the two previous equations we have also added the contribution, coming from the tree diagrams, that contains the bare coupling constant. In an ultraviolet finite theory as string theory we should not deal with a bare and a renormalized coupling. On the other hand, we have already discussed the fact that the introduction of a gauge field produces a string amplitude that is already divergent at the string level and that therefore must be regularized with the introduction of a cutoff.

We have already mentioned that Eq.s (51) and (52) are equal to each other as one can see by performing the modular transformation $\tau = \frac{1}{t}$. Actually one can see that, under such a transformation, the contribution of the massless open string states gets transformed into that of the massless closed string states and viceversa. This follows from the fact that the threshold corrections vanish in the two channels. This also means that the open/closed string duality exactly maps the ultraviolet divergent contribution coming from the massless open string states circulating in the loop and that reproduces the divergences of $\mathcal{N}=2$ super Yang-Mills, living on the world-volume of the fractional D3 branes, into the infrared divergent contribution due to the massless closed string states propagating between the two boundary states.

In the open string channel the integrals are naturally regularized in the infrared ($\tau \to \infty$) by the fact that the two stacks of branes are at a finite distance y. In the closed string channel the presence of a distance y between the branes makes the integral convergent in the ultraviolet ($t \to 0$), but instead an infrared cutoff Λ is needed. If we identify the two cutoffs Λ's, we see that the expressions in the two field theory limits are actually equal! This observation clarifies now why the supergravity solution gives the correct answer for the perturbative behaviour of the non-conformal world-volume theory as found in Ref.s [51, 52, 54, 53, 49] and reviewed in Ref. [3]. In fact, we can extract the coefficient of the term F^2 from either of the two Eq.s (51) and (52) getting the following expression:

$$\frac{1}{g^2_{YM}(\epsilon)} + \frac{N}{8\pi^2}\log\frac{y^2}{\epsilon^2} \equiv \frac{1}{g^2_{YM}(y)} \quad , \quad \epsilon^2 \equiv 2\pi(\alpha'\Lambda)^2 \quad , \tag{53}$$

where the integral appearing in Eq. (51) has been explicitly computed:

$$I(\Lambda, y) \equiv \int_{1/\alpha'\Lambda^2}^{\infty} \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi(\alpha')}} \simeq \log\frac{2\pi(\alpha'\Lambda)^2}{y^2} \quad . \tag{54}$$

Eq. (53) is equal to the first equation in (23) for $M=0$ where we identify the square of the bare coupling constant $g^2_{YM}(\Lambda)$ computed at the cutoff $\Lambda \sim \epsilon/\alpha'$ with $8\pi g_s$. The previous derivation makes it clear why the running coupling constant of $\mathcal{N}=2$ super Yang-Mills can be obtained from the supergravity solution corresponding to N fractional D3 branes of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$.

Eq. (53) gives the one-loop correction to the bare gauge coupling constant $g_{YM}(\Lambda)$ in the gauge theory regularized with the cutoff Λ. The renormalization procedure can then be performed by introducing the renormalized coupling constant $g^{ren}_{YM}(\mu)$ given in terms of the bare one by the relation:

$$\frac{1}{g^2_{YM}(\Lambda)} = \left(\frac{1}{g^2_{YM}(\mu)}\right)^{ren} + \frac{N}{8\pi^2}\log\frac{\Lambda^2}{\mu^2} \quad , \tag{55}$$

with μ being the renormalization scale. Using the previous equation in Eq. (53) one can rewrite the coefficient of the F^2 term in terms of the renormalized gauge coupling constant

$$\left(\frac{1}{g_{YM}^2(\mu)}\right)^{ren} + \frac{N}{8\pi^2}\log\frac{m^2}{\mu^2} = \left(\frac{1}{g_{YM}^2(m)}\right)^{ren} \quad ; \quad m^2 \equiv \frac{y^2}{2\pi\alpha'^2} \quad . \tag{56}$$

From it, or equivalently from Eq. (55), we can now determine the one-loop β-function:

$$\beta \equiv \mu\frac{\partial}{\partial\mu}g_{YM}^{ren}(\mu) = -\frac{g_{YM}^{ren\,3}N}{8\pi^2} \quad , \tag{57}$$

that is the correct one for $\mathcal{N}=2$ super Yang-Mills.

Let us turn now to the vacuum angle θ_{YM} that is provided by the terms in Eq.s (51) and (52) with the topological charge. If we extract it from either of the two Eq.s (51) and (52) we find that it is imaginary and moreover must be renormalized as the coupling constant. A way of eliminating these problems is to introduce a complex cutoff, to allow the gauge field to be in either one of the two stacks and by taking the symmetric combination:

$$\frac{1}{2}\left[\langle D3;F|D|D3\rangle + \langle D3|D|D3;F\rangle\right] = \frac{1}{2}\left[\langle D3;F|D|D3\rangle + \overline{\langle D3;F|D|D3\rangle}\right] \quad . \tag{58}$$

If we introduce a complex cutoff $\Lambda \to \Lambda e^{-i\theta}$ the divergent integral in Eq. (51) becomes:

$$I(z) \equiv \int_{1/(\alpha'\Lambda^2 e^{-2i\theta})}^{\infty}\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \sim \log\frac{2\pi(\alpha'\Lambda)^2}{y^2e^{2i\theta}} = \log\frac{2\pi(\alpha'\Lambda)^2}{z^2} \quad . \tag{59}$$

This procedure leaves unchanged all the previous considerations concerning the gauge coupling constant because in this case one gets as before:

$$\frac{1}{2}\left[I(z) + I(\bar z)\right] = \log\frac{2\pi(\alpha')^2\Lambda^2}{y^2} \quad . \tag{60}$$

For the θ_{YM} angle one gets instead:

$$\theta_{YM} = -i\frac{N}{2}\left[I(z) - I(\bar z)\right] = -2N\theta \quad , \tag{61}$$

that exactly reproduces the result given in the second equation in (23) for $M=0$. Remember, however, that in Eq. (23) θ is the phase of the complex quantity $z = ye^{i\theta}$. But, as it can be seen in Eq. (59), giving a phase to the cutoff corresponds to give the opposite phase to the distance y between the branes. We prefer to complexify the cutoff rather than y in order to keep the open string Virasoro operator L_0 real, but the effect is in fact the same.

In the second part of this subsection we consider a bound state of N D3 branes and M D7 branes in order to add matter hypermultiplets to the pure gauge theory considered until now. In particular we consider the same brane configuration as the one analyzed in the previous subsection.

Let us start by giving the spectrum of the massless open strings stretched between the two stacks of D3 and D7 branes. Physical states are taken in the picture -1 for the NS

sector and in the picture $-1/2$ for the R one. The massless ones come from the lowest level and are given by:

$$\lambda_{\rm NS}|0;\, k\rangle_{-1}\otimes|s_0=1/2,\, s_3,\, s_4\rangle \qquad \lambda_{\rm R}|s_0=1/2,\, s_1,\, s_2\rangle_{-1/2}\otimes|0;\, k\rangle \quad . \tag{62}$$

The structure of the states in Eq. (62) can be easily understood if one considers that the orbifold projection changes the modding of the oscillators along the four directions spanned by the orbifold. Moreover the orbifold breaks the Lorentz group $SO(1,9)$ to $SO(1,5)\otimes SO(4)$ and in Eq. (62) we have correspondingly splitted the string ground states. In Eq.s (62) λ's denote the Chan-Paton factors and s_i ($i=1,\ldots,4$) are the eigenvalues of the "number operators" N_i so defined[8] :

$$\Gamma^0\Gamma^1=-2N_0 \quad , \quad \Gamma^{2i}\Gamma^{2i+1}=2\,i\,N_i \qquad \text{with } i=1,\ldots,4 \quad . \tag{63}$$

We remind that $s_0=1/2$ and that the GSO projectors are defined as:

$$P^{\rm NS}_{GSO}=\frac{(-1)^{G_{\beta\gamma}}-\Gamma^6\ldots\Gamma^9(-1)^F}{2}=\frac{(-1)^{G_{\beta\gamma}}+2^2N_3N_4(-1)^F}{2} \tag{64}$$

and

$$P^{\rm R}_{GSO}=\frac{(-1)^{G_{\beta\gamma}}-\Gamma^0\ldots\Gamma^5(-1)^{F_R}}{2}=\frac{(-1)^{G_{\beta\gamma}}-2^3N_0N_1N_2(-1)^{F_R}}{2} \tag{65}$$

where the fermion numbers F and F_R are obtained from the corresponding ones defined in Eq.s (38) and (39) by changing modding of the oscillators along the mixed $6\ldots9$ directions, i.e. in the NS sector the modding of the fermionic [bosonic] oscillators is integer [half-integer] while in the R one is half-integer [half-integer].

Eq.s (64) and (65) impose that $s_1=-s_2$ and $s_3=-s_4$. In the NS sector we have two real scalars while in the R sector we have two Weyl fermions. Altogether they form an hypermultiplet.

The physical states are the ones left invariant by the orbifold. In the R sector the non trivial generator h of the orbifold group acts as:

$$\begin{gathered}\lambda^{\rm R}_{ij}|s_0=1/2,\, s_1,\, s_2\rangle_{-1/2}\otimes|0,\, k\rangle\rightarrow\\ \left(\gamma^{\rm D3}_h\right)_{ih}\lambda^{\rm R}_{hk}(\gamma^{\rm D7}_h)^{-1}_{kj}|s_0=1/2,\, s_1,\, s_2\rangle_{-1/2}\otimes|0;\, k\rangle \qquad ,\end{gathered} \tag{66}$$

where γ_h is the orbifold action on the Chan-Paton factors and can be taken as $\pm\mathbb{I}$ depending on the kind of fractional brane considered. The previous expression implies that the surviving open strings are those stretched between fractional branes of the same kind for which one has $\gamma^{\rm D3}_h=\gamma^{\rm D7}_h$, while strings stretched between different kinds of branes are projected out. Analogous considerations hold in the NS sector where the non trivial element h of the orbifold group acts on the oscillator vacuum as $h=-4N_3N_4$.

[8] In this paper the definition of number operators differs by a factor 2 from the corresponding expressions given in Ref.s [38, 17] and, as previously noticed, N_0 has an opposite sign.

The annulus diagram for open strings stretched between a dressed D3 and a bunch of M D7 branes is given, as in the previous case, by two contributions, $Z_{37} = Z^o_{e;37} + Z^o_{h;37}$, with:

$$
\begin{aligned}
Z^o_{e;37} &= -\frac{M}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{2\sin\pi\nu_f\, 2\sin\pi\nu_g}{\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)\Theta_4^2(0|i\tau)} \\
&\times \frac{1}{4}\left[\Theta_2^2(0|i\tau)\Theta_3(i\tau\nu_g|i\tau)\Theta_3(i\tau\nu_f|i\tau) - \Theta_3^2(0|i\tau)\Theta_2(i\tau\nu_g|i\tau)\Theta_2(i\tau\nu_f|i\tau)\right] \\
&+ i\frac{M}{4}\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau} e^{-\frac{y^2}{2\pi\alpha'}}
\end{aligned}
\tag{67}
$$

and

$$
\begin{aligned}
Z^o_{h;37} &= \mp\frac{M}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{2\,\sin\pi\nu_f 2\,\sin\pi\nu_g}{\Theta_3^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \\
&\times \frac{1}{4}\left[\Theta_2^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \Theta_4^2(0|i\tau)\Theta_2(i\nu_f\tau|i\tau)\Theta_2(i\nu_g\tau|i\tau)\right] \\
&\pm i\frac{M}{4}\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau} e^{-\frac{y^2}{2\pi\alpha'}} \quad .
\end{aligned}
\tag{68}
$$

In the latter equation the upper [lower] sign refers to the case of open strings stretched between fractional branes of the same [different] kind.

In Eq. (67) the first and the second term come respectively from the NS and R sector, while the last one comes from the R$(-1)^F$ sector. The $NS(-1)^F$ sector does not contribute due to the vanishing of the trace over the zero modes. The first two terms of Eq. (67) can be obtained respectively from the first and the third term in Eq. (40) by changing, as explained after Eq. (65), the modding of the string oscillators along the directions $6\ldots 9$. Such a modification transforms $\Theta_3(0|i\tau)$ into $\Theta_2(0|i\tau)$, leaving unchanged the Θ's having in their argument ν_f or ν_g, and maps the NS and R terms of Eq. (40) into the first two terms of Eq. (67). Analogous considerations can be extended to Eq. (68) where the first, the second and the third terms correspond respectively to the NS$(-1)^F$, R and R$(-1)^F$ sector. The first term of Eq. (68) can be obtained from the second term in Eq. (41) with the substitution $\Theta_3 \to \Theta_2$ as before. The second term of Eq. (68) is obtained from the first one in Eq. (41) by changing first $\Theta_3 \to \Theta_2$ and $\Theta_4 \to \Theta_1$ which map the NS in the R sector and then by performing the transformation $\Theta_1(0|i\tau) \to \Theta_4(0|i\tau)$ that corresponds to change modding along the mixed directions.

Eq. (67) can be rewritten in a more compact form by using Eq. (372) with:

$$
g_i = h_3 = h_4 = 0, \quad h_1 = -h_2 = 1, \quad \nu_1 = \nu_2 = 0, \quad \nu_3 = i\tau\nu_f, \quad \nu_4 = i\tau\nu_g \ . \tag{69}
$$

In this way one gets the following identity:

$$
\begin{aligned}
&\Theta_2^2(0|i\tau)\Theta_3(i\tau\nu_g|i\tau)\Theta_3(i\tau\nu_f|i\tau) - \Theta_3^2(0|i\tau)\Theta_2(i\tau\nu_g|i\tau)\Theta_2(i\tau\nu_f|i\tau) = \\
&\Theta_4^2(0|i\tau)\Theta_1(i\tau\nu_f|i\tau)\Theta_1(i\tau\nu_g|i\tau) + 2\Theta_4^2(i\frac{\tau}{2}(\nu_f+\nu_g)|i\tau)\Theta_1^2(i\frac{\tau}{2}(\nu_f-\nu_g)|i\tau) \ ,
\end{aligned}
\tag{70}
$$

which allows us to rewrite Eq. (67) as:

$$
\begin{aligned}
Z^{o}_{e;37} &= -\frac{M}{(8\pi^2\alpha')^2}\int d^4x\sqrt{\det\left(\eta+\hat{F}\right)}\sin\pi\nu_f\sin\pi\nu_g\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \\
&\times\left\{1+2\frac{\Theta_4^2(i\frac{\tau}{2}(\nu_f+\nu_g)|i\tau)\Theta_1^2(i\frac{\tau}{2}(\nu_f-\nu_g)|i\tau)}{\Theta_4^2(0|i\tau)\Theta_1(i\tau\nu_f|i\tau)\Theta_1(i\tau\nu_g|i\tau)}\right\} \\
&+i\frac{M}{4}\left[\frac{1}{32\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{1/(\alpha'\Lambda^2)}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2}{2\pi\alpha'}}\quad . \qquad (71)
\end{aligned}
$$

Analogously, by using again Eq. (372), but this time with:

$$
\begin{aligned}
&h_3=-h_4=g_3=-g_4=1 \qquad h_1=h_2=g_1=g_2=0 \\
&\nu_1=i\nu_f\tau \qquad \nu_2=i\nu_g\tau \qquad \nu_3=\nu_4=0\quad , \qquad (72)
\end{aligned}
$$

one gets the following identity:

$$
\begin{aligned}
&\Theta_2^2\left(0|i\tau\right)\Theta_4\left(i\nu_f\tau|i\tau\right)\Theta_4\left(i\nu_g\tau|i\tau\right)-\Theta_4^2\left(0|i\tau\right)\Theta_2\left(i\nu_f\tau|i\tau\right)\Theta_2\left(i\nu_g\tau|i\tau\right) \\
&\quad=\Theta_3^2\left(0|i\tau\right)\Theta_1\left(i\nu_f\tau|i\tau\right)\Theta_1\left(i\nu_g\tau|i\tau\right) \\
&\quad+2\Theta_1^2\left(i\tfrac{\tau}{2}(\nu_f-\nu_g)|i\tau\right)\Theta_3^2\left(i\tfrac{\tau}{2}(\nu_f+\nu_g)\tau|i\tau\right)\quad , \qquad (73)
\end{aligned}
$$

which allows one to write Eq. (68) as follows:

$$
\begin{aligned}
Z^{o}_{h;37} &= \mp\frac{M}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\sin\pi\nu_f\sin\pi\nu_g\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \\
&\times\left\{1+2\frac{\Theta_1^2\left(i\frac{\tau}{2}(\nu_f-\nu_g)|i\tau\right)\Theta_3^2\left(i\frac{\tau}{2}(\nu_f+\nu_g)|i\tau\right)}{\Theta_3^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\right\} \\
&\pm i\frac{M}{4}\left[\frac{1}{32\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{1/(\alpha'\Lambda^2)}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2}{2\pi\alpha'}}\quad . \qquad (74)
\end{aligned}
$$

Expanding Eq.s (71) and (74) up to the quadratic terms in the gauge field, according to the procedure above explained, yields:

$$
\begin{aligned}
Z^{o}_{h;37}=\pm Z^{o}_{e;37} &= \mp\frac{M}{2(8\pi^2\alpha')^2}\int d^4x[2\,if\,g+(if-g)^2]\int_{1/(\alpha'\Lambda^2)}^\infty\frac{d\tau}{\tau}e^{-y^2\tau/(2\pi\alpha')} \\
&\pm\ i\frac{M}{4}\left[\frac{1}{32\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_{1/(\alpha'\Lambda^2)}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2}{2\pi\alpha'}} \\
&= \pm\frac{M}{2(4\pi)^2}\int d^4x\left[-\frac{1}{4}\left(F^a_{\mu\nu}F^{a\,\mu\nu}-iF^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right)\right] \\
&\quad\times\int_{1/(\alpha'\Lambda^2)}^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}. \qquad (75)
\end{aligned}
$$

From it we can extract the gauge coupling constant:

$$
\begin{aligned}
\frac{1}{g^2_{YM}} &= \frac{M}{(4\pi)^2}\left(\frac{1}{2}\pm\frac{1}{2}\right)\int_{1/(\alpha'\Lambda^2)}^\infty\frac{d\tau}{\tau}e^{-y^2\tau/(2\pi\alpha')} \\
&= \frac{M}{(4\pi)^2}\left(\frac{1}{2}\pm\frac{1}{2}\right)\log\frac{2\pi(\alpha'\Lambda)^2}{y^2} \qquad (76)
\end{aligned}
$$

and by taking, as explained in Eq. (58), the symmetric combination, we obtain for the $\theta_{\rm YM}$ angle:

$$\theta_{\rm YM} = \left(\frac{1}{2} \pm \frac{1}{2}\right) M\theta \quad , \tag{77}$$

where the positive [negative] sign corresponds to open strings stretched between fractional branes of the same [different] kind.

These are just the expected running coupling constant and chiral anomaly and, when added respectively to Eq. (53) and (61), coincide with Eq. (23) obtained via supergravity. It is important to observe that also in this case threshold corrections vanish and this provides the reason why the contribution of the massless string states gets transformed into that of the massless closed string states and viceversa, making the gauge/gravity correspondence to hold.

3.3 Gauge/gravity correspondence from open/closed duality: $\mathcal{N} = 1$ case

In the following we extend the analysis performed in the previous section to the case of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ that preserves four supersymmetry charges. The orbifold group $\mathbb{Z}_2\times\mathbb{Z}_2$ contains four elements whose action on the three complex coordinates:

$$z_1 = x_4 + ix_5 \qquad z_2 = x_6 + ix_7 \qquad z_3 = x_8 + ix_9 \tag{78}$$

is chosen to be:

$$R_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \qquad R_{h_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} ,$$
$$R_{h_2} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} , \qquad R_{h_3} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{79}$$

The orbifold group acts also on the Chan-Paton factors. Fractional branes are defined as branes for which these latter transform according to irreducible representations of the orbifold group. $\mathbb{Z}_2\times\mathbb{Z}_2$ has four irreducible one-dimensional representations that correspond to four different kinds of fractional branes. The orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, as already explained in Ref.s [62, 70], can be seen as obtained by three copies of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ where the i-th $\mathbb{Z}_2$ contains the elements (e, h_i) $(i = 1, 2, 3)$. This means that the boundary states associated to each fractional brane are:

$$\begin{aligned}
|Dp>_1 &= |Dp>_u + |Dp>_{t_1} + |Dp>_{t_2} + |Dp>_{t_3} \; , \\
|Dp>_2 &= |Dp>_u + |Dp>_{t_1} - |Dp>_{t_2} - |Dp>_{t_3} \; , \\
|Dp>_3 &= |Dp>_u - |Dp>_{t_1} + |Dp>_{t_2} - |Dp>_{t_3} \; , \\
|Dp>_4 &= |Dp>_u - |Dp>_{t_1} - |Dp>_{t_2} + |Dp>_{t_3} \; ,
\end{aligned} \tag{80}$$

where $|Dp>_u$ is the untwisted boundary state that, apart from an overall factor $\frac{1}{2}$ due to the orbifold projection, is the same as the one in flat space and $|Dp>_{t_i}$ $(i = 1, 2, 3)$ are exactly the same as the twisted boundary states on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, apart from a factor $\frac{1}{\sqrt{2}}$. The signs in front of each twisted term in Eq. (80) depend on the irreducible representation chosen for the orbifold group action on the Chan-Paton factors.

In order to keep the forthcoming discussion as general as possible, we study the one-loop vacuum amplitude of an open string stretching between a stack of N_I $(I = 1, \ldots, 4)$ branes of type I and a D3 fractional brane, for example of type $I = 1$, with a background $SU(N)$ gauge field turned-on on its world-volume. Due to the structure of the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$, this amplitude is the sum of four terms:

$$Z = Z_e + \sum_{i=1}^{3} Z_{h_i}, \tag{81}$$

where Z_e and Z_{h_i} are obtained in the open [closed] channel by multiplying Eq.s (40) and (41) [Eq.s (45) and (46)] by an extra 1/2 factor due to the orbifold projection. Since, even in this case, we are interested in analyzing the meaning of the divergences in the non-trivial twisted contributions Z_{h_i}, we focus on these latter. In particular, in the open string channel, $Z^o_{h_i}$ is:

$$\begin{aligned} Z^o_{h_i} &= \frac{f_i(N)}{2\,(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \frac{2\sin\pi\nu_f 2\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \\ &\times \left\{\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right\} \\ &- \frac{i\,f_i(N)}{64\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int\frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \quad . \end{aligned} \tag{82}$$

In the previous expression we should put to zero the distance y_i between the stack of the N_I branes and the dressed one, since the fractional branes are constrained to live at the orbifold fixed point $z_1 = z_2 = z_3 = 0$. However y_i provides a natural infrared cut-off in Eq. (82). Therefore we keep this quantity small but finite and we are going to put it to zero just at the end of the calculation. The functions $f_i(N)$ introduced in Eq. (82) depend on the number of the different kinds of fractional branes N_I and their explicit expressions are: [62, 70]

$$\begin{aligned} f_1(N_I) &= N_1 + N_2 - N_3 - N_4 \quad , \\ f_2(N_I) &= N_1 - N_2 + N_3 - N_4 \quad , \\ f_3(N_I) &= N_1 - N_2 - N_3 + N_4 \quad . \end{aligned} \tag{83}$$

Let us now extract in both channels the quadratic terms in the gauge field F. In the open sector, we get:

$$\begin{aligned} Z^o_h(F) \quad &\rightarrow \quad \left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \sum_{i=1}^{3}\frac{f_i(N)}{16\pi^2}\left[\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}}\right]\right\} \\ &-i\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\sum_{i=1}^{3}\frac{f_i(N)}{2}\int_{\frac{1}{\alpha'\Lambda^2}}^\infty \frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \quad , \end{aligned} \tag{84}$$

while in the closed string channel we obtain:

$$
\begin{aligned}
Z_h^c(F) \quad \rightarrow \quad & \left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \sum_{i=1}^{3}\frac{f_i(N)}{16\pi^2}\left[\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y_i^2}{2\pi\alpha' t}}\right]\right\} \\
& -i\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\sum_{i=1}^{3}\frac{f_i(N)}{2}\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y_i^2}{2\pi\alpha' t}} \quad . \qquad (85)
\end{aligned}
$$

Analogously to the case of the previous orbifold the divergent contribution is due to the massless states in both channels. We have also introduced the one coming from the tree diagrams. The main properties exhibited by the interactions in the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold, given in Eq.s (51) and (52), are also shared by the interactions in the $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ orbifold, given in Eq.s (84) and (85). In particular, also in this case, one can see that the contribution of the massive states vanishes and the open string massless contribution is transformed into the closed string massless one. This confirms the main result obtained in the previous subsection, i.e. the open/closed string duality exactly maps the ultraviolet divergent contribution coming from the massless open string states, which reproduces the divergences of $\mathcal{N}=1$ super Yang-Mills, into the infrared divergent contribution due to the massless closed string states. By extracting the coefficient of the term F^2 in Eq. (84) or Eq. (85), we get:

$$
\frac{1}{g^2_{\rm YM}(\epsilon)} + \frac{1}{16\pi^2}\sum_{i=1}^{3} f_i(N_I)\log\frac{y_i^2}{\epsilon^2} \equiv \frac{1}{g^2_{\rm YM}}(y) \qquad \epsilon^2 = 2\pi(\alpha'\Lambda)^2 \quad . \qquad (86)
$$

Eq. (86) reproduces Eq. (3.14) of Ref. [12] clarifying again why the supergravity solution, dual to $\mathcal{N}=1$ super Yang-Mills theory, gives the correct answer for the perturbative behaviour of the non-conformal world-volume theory, as found in Ref.s [12, 62, 70].

In performing the renormalization procedure, we introduce the renormalized coupling constant $g^{ren}_{\rm YM}(\mu)$ given in terms of the bare one by the relation:

$$
\begin{aligned}
\frac{1}{g^2_{YM}(\Lambda)} &= \left(\frac{1}{g^2_{YM}(\mu)}\right)^{ren} + \sum_{i=1}^{3}\frac{f_i(N)}{16\pi^2}\log\frac{\Lambda^2}{\mu^2} \\
&= \left(\frac{1}{g^2_{YM}(\mu)}\right)^{ren} + \frac{3N_1 - N_2 - N_3 - N_4}{16\pi^2}\log\frac{\Lambda^2}{\mu^2} \quad . \qquad (87)
\end{aligned}
$$

From this equation we can obtain the β-function:

$$
\beta(g^{ren}_{\rm YM}) \equiv \mu\frac{\partial}{\partial\mu}g^{ren}_{YM}(\mu) = -\frac{g^{ren\,3}_{YM}}{16\pi^2}\left(3N_1 - N_2 - N_3 - N_4\right) \quad , \qquad (88)
$$

that is the correct one for the world-volume theory living on the dressed brane.

Finally, in the same spirit as in Sect. 3, we consider the symmetric combination given in Eq. (58) and by introducing a complex cut-off $\Lambda e^{-i\theta}$, or equivalently the complex coordinates $z_i = y_i e^{i\theta}$, we get the following expression for $\theta_{\rm YM}$:

$$
\theta_{\rm YM} = -\sum_{i=1}^{3} f_i(N_I)\theta \quad , \qquad (89)
$$

that is again in agreement with the result given in Eq. (3.14) of Ref. [12].

4 Gauge/Gravity Correspondence in Bosonic String Theory

In this section we study the validity of the gauge/gravity correspondence in the 26-dimensional bosonic string and in order to compare it with the supersymmetric case discussed in the previous section, we consider it in the orbifold $C^{\delta/2}/\mathbb{Z}_2$ with $\delta < 22$. As in the previous section we consider the one-loop vacuum amplitude of an open string stretching between a D3 brane dressed with a background gauge field and a system of N undressed D3 branes. It is given by:

$$Z = N \int_0^\infty \frac{d\tau}{\tau} Tr \left[\left(\frac{e+h}{2} \right) (-1)^{G_{bc}} e^{-2\pi\tau L_0} \right] \equiv \ Z_e^o + Z_h^o \ , \tag{90}$$

where L_0 includes the ghost and the matter contribution, the first one having the same structure as in flat space, while the matter part being derived in Appendix B of Ref. [17]. By performing the explicit calculation of the one-loop vacuum amplitude one gets:

$$Z_e^o = -\frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \\ \times \frac{2 e^{\pi\tau(\nu_f^2 + \nu_g^2)} \sin\pi\nu_f \sin\pi\nu_g}{f_1^{18}(e^{-\pi\tau}) \Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \tag{91}$$

and

$$Z_h^o = -\frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \left[\frac{2 e^{\pi\tau(\nu_f^2 + \nu_g^2)} \sin\pi\nu_f \sin\pi\nu_g}{\Theta_1(i\tau\nu_f|i\tau)\Theta_1(i\tau\nu_g|i\tau)} \right] \\ \times 2^{\frac{\delta}{2}} [f_1(k)]^{-(18-\delta)} [f_2(k)]^{-\delta}, \tag{92}$$

where the power 18 is obtained from $d-8$ for the value of the critical dimension $d=26$. The calculation of the untwisted sector was originally performed in Ref. [56] for the case of D9 branes.

The previous expressions can also be rewritten in the closed string channel and one gets:

$$Z_e^c = \frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^{11}} e^{-\frac{y^2}{2\pi\alpha' t}} \frac{2\sin\pi\nu_f \sin\pi\nu_g}{f_1^{18}(e^{-\pi t})\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)} \tag{93}$$

for the untwisted sector and

$$Z_h^c = \frac{N}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^{11-\delta/2}} e^{-\frac{y^2}{2\pi\alpha' t}} \left[\frac{2\sin\pi\nu_f \sin\pi\nu_g}{\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)} \right] \\ \times 2^{\delta/2} [f_1(q)]^{-(18-\delta)} [f_4(q)]^{-\delta} \tag{94}$$

for the twisted sector. They can be shown to be equal to Eq.s (91) and (92) respectively by using Eq.s (359) and (364) for Θ_1, f_1 and f_2.

Eq. (392) and Eq.s (394) allow one to extract easily from Eq. (90) the coefficient of the kinetic term for the gauge field that turns out to be:

$$\frac{1}{g_{YM}^2} = -\frac{N}{2(4\pi)^2} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{(2\pi\alpha')}} \left[\frac{1}{3\tau^2} + \frac{1}{\pi\tau} - 2k \frac{d}{dk} \log f_1(k) \right] f_1^{-24}(k) \\ \times \left[1 + 2^{\frac{\delta}{2}} \left(\frac{f_1(k)}{f_2(k)} \right)^\delta \right]. \tag{95}$$

The previous expression gives the running coupling constant including all the threshold corrections coming from the massive open string states. Eq. (95) can be written more explicitly in the following form:

$$\frac{1}{g_{YM}^2} = -\frac{N}{2(4\pi)^2}\int_0^\infty \frac{d\tau}{\tau} e^{-y^2\tau/(2\pi\alpha')}\left[k^{-2}\prod_{n=1}^{\infty}(1-k^{2n})^{-24}\right]\times$$

$$\times\left[\frac{1}{3\tau^2}+\frac{1}{\pi\tau}-\frac{1}{6}+4\sum_{n=1}^{\infty}\frac{k^{2n}}{(1-k^{2n})^2}\right]\left[1+\prod_{n=1}^{\infty}\left(\frac{1-k^{2n}}{1+k^{2n}}\right)^{\delta}\right], \tag{96}$$

where we have used Eq. (395). Notice that, differently from the supersymmetric case, the threshold corrections to the gauge kinetic term do not vanish. Indeed, from Eq. (96) one can see that both the massless and massive bosonic open string states contribute to the gauge coupling constant. The β-function of the gauge theory living on the stack of branes can be computed by selecting only the contribution of the massless states. This can be done by performing the field theory limit ($\alpha'\tau \equiv \sigma$ fixed with $\alpha' \to 0$ and $\tau \to \infty$). In this way from Eq. (96) one gets:

$$\frac{1}{g_{YM}^2} = -\frac{N}{2(4\pi)^2}\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau} e^{-y^2\tau/(2\pi\alpha')}\left[\frac{1+24k^2}{k^2}\left(-\frac{1}{6}+4k^2\right)(2-2\delta k^2)+\dots\right]$$

$$\simeq -\frac{N}{16\pi^2}\frac{\delta}{6}\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} = -\frac{N}{(4\pi)^2}\frac{\delta}{6}\log\frac{2\pi(\alpha'\Lambda)^2}{y^2} \quad , \tag{97}$$

where in the last step we have neglected the term proportional to k^{-2} corresponding to the tachyon and used Eq. (54). Adding finally the contribution from the tree diagrams yields:

$$\frac{1}{g_{YM}^2} = \frac{1}{g_{YM}^2(\Lambda)} + \frac{N}{16\pi^2}\frac{\delta}{6}\log\frac{y^2}{\epsilon^2} \quad ; \quad \epsilon^2 \equiv 2\pi(\alpha'\Lambda)^2 \quad . \tag{98}$$

Notice that the contribution of massless states in Eq. (97) coming from the untwisted sector vanishes, in agreement with the fact that it corresponds to the one-loop β-function coefficient of a gauge theory with one gluon and 22 scalars [9]. One gets instead a non-zero contribution from the twisted sector that reproduces the right one-loop β-function for a gauge theory with one vector field and $N_s = 22-\delta$, (i.e. the number of directions orthogonal both to the D3 brane and to the orbifold) scalars :

$$\beta(g_{YM}) = \frac{g_{YM}^3}{(4\pi)^2}\left[-\frac{11}{3}+\frac{N_s}{6}\right] = -\frac{\delta}{6}\frac{g_{YM}^3}{(4\pi)^2} \quad . \tag{99}$$

Notice that in the bosonic case there are no terms proportional to the topological charge coming from Eq.s (93) and (94), consistently with the fact that the world-volume theory is a purely bosonic gauge theory and therefore not affected by chiral anomaly.

[9] See Ref. [68] and Appendix B of Ref. [16].

One can also extract the coefficient of the gauge kinetic term from the amplitude written in the closed string channel in Eq.s (93) and (94). Eq.s (394) and (398) must be used, obtaining:

$$
\begin{aligned}
\frac{1}{g_{YM}^2} &= -\frac{N}{2(4\pi)^2}\int_0^\infty \frac{dt}{t^{11}} e^{-\frac{y^2}{(2\pi\alpha' t)}} \left[\frac{1}{3} + 2q\frac{d}{dq}\log f_1(q)\right] f_1^{-24}(q) \\
&\qquad \times \left[1 + (2t)^{\delta/2}\left(\frac{f_1(q)}{f_4(q)}\right)^{\delta}\right] . \qquad (100)
\end{aligned}
$$

Using the modular transformations of the various functions f_i and the relation:

$$
\tau^2 k \frac{d}{dk} = -q\frac{d}{dq} \qquad (101)
$$

that implies

$$
-2k\frac{d}{dk}\log f_1(k) + \frac{1}{\pi\tau} = 2t^2 q\frac{d}{dq}\log f_1(q) \quad , \qquad (102)
$$

it is easy to see that Eq.s (95) and (100) transform into each other. It is convenient to write Eq. (100) in the following more explicit way:

$$
\frac{1}{g_{YM}^2} = -\frac{N}{2(4\pi)^2}\int_0^\infty \frac{dt}{t^{11}} e^{-\frac{y^2}{(2\pi\alpha' t)}} q^{-2}\prod_{n=1}^\infty (1-q^{2n})^{-24}
$$

$$
\times \left[\frac{1}{2} - \sum_{n=1}^\infty \frac{4nq^{2n}}{1-q^{2n}}\right]\left[1 + (2t)^{\delta/2} q^{\delta/8}\prod_{n=1}^\infty\left(\frac{1-q^{2n}}{1-q^{2n-1}}\right)^{\delta}\right] , \qquad (103)
$$

where we have used Eq. (395). As in the open channel, we find that there are threshold corrections to the running coupling constant due to massive closed string states.

In order to understand the role of massless closed string states, let us first examine the mass spectrum of closed strings, which is given by:

$$
\frac{\alpha'}{2}M^2 = N + \tilde{N} - 2 + \frac{\delta}{8} \quad , \quad N = \tilde{N} \quad , \qquad (104)
$$

where

$$
N = \sum_{n=1}^\infty \left[n a_n^\dagger \cdot a_n + (n - \frac{1}{2}) a_{n-1/2}^\dagger \cdot a_{n-1/2}\right] , \qquad (105)
$$

with the analogous expression for $\tilde{N}$ and $\delta = 0$ ($\delta \neq 0$) in the untwisted (twisted) sector. The intercept in Eq. (104) comes from the zero-point energy:

$$
2\frac{24-\delta}{2}\sum_{n=1}^\infty n + 2\frac{\delta}{2}\sum_{n=1}^\infty \frac{2n-1}{2} = -\frac{24-\delta}{12} + \frac{\delta}{24} = -2 + \frac{\delta}{8} \quad . \qquad (106)
$$

The zero-point energy derived in the previous equation matches with the power of q appearing in Eq. (103) both for the twisted and the untwisted sector.

By performing the field theory limit ($t \to \infty, \alpha' \to 0$ and $\alpha' t$ fixed) and neglecting the divergent contribution due to the closed string tachyon, one gets a vanishing result in the

untwisted sector. In the twisted sector one has again a tachyon in the spectrum, if $N_s > 6$, that becomes massless if $N_s = 6$ and massive if $N_s < 6$.

Even in the case $N_s = 6$ ($\delta = 16$) in which the spectrum admits massless states, by taking the field theory limit one gets again a vanishing result, not reproducing Eq. (98) from the closed channel. From the previous analysis it follows that the gauge/gravity correspondence does not hold in the present case.

5 Gauge/Gravity Correspondence in Type 0B String Theory

In this section we summarize the properties of type 0 string theories [73] and explore the gauge/gravity correspondence in this framework. In particular we discuss the spectrum and D branes of type 0B theory and compute the annulus diagram. Finally, we consider the one-loop vacuum amplitude of an open string stretching between a stack of D branes and a brane having an external field on its world-volume, and we study under which conditions the gauge/gravity correspondence holds.

Type 0 string theories are non-supersymmetric closed string models obtained by applying the following non-chiral diagonal projections on the Neveu-Scharz-Ramond model:

$$P_{\mathrm{NS-NS}} = \frac{1 + (-1)^{F+\tilde{F}+G_{\beta\gamma}+\tilde{G}_{\beta\gamma}}}{2} \qquad P_{\mathrm{R-R}} = \frac{1 \pm (-1)^{F+\tilde{F}+G_{\beta\gamma}+\tilde{G}_{\beta\gamma}}}{2}, \tag{107}$$

where the upper [lower] sign in $P_{\mathrm{R-R}}$ corresponds to 0B [0A]. $G_{\beta\gamma}$ is defined in Eq.s (37), F is the world-sheet fermion number defined in Eq.s (38) and (39) with analogous definitions for $\tilde{F}$ and $(-1)^{\tilde{F}}$. In addition it is imposed that the fermionic NS-R and R-NS sectors are eliminated from the physical spectrum, obtaining a purely bosonic string model.

5.1 Closed string spectrum

The closed string spectrum can be determined by keeping only the string states that are left invariant by the action of the operators given in Eq. (107). It results to be:

$$\text{type 0A} \qquad (\mathrm{NS}-,\ \mathrm{NS}-) \otimes (\mathrm{NS}+,\ \mathrm{NS}+) \otimes (\mathrm{R}-,\ \mathrm{R}+) \otimes (\mathrm{R}+,\ \mathrm{R}-) \tag{108}$$

$$\text{type 0B} \qquad (\mathrm{NS}-,\ \mathrm{NS}-) \otimes (\mathrm{NS}+,\ \mathrm{NS}+) \otimes (\mathrm{R}-,\ \mathrm{R}-) \otimes (\mathrm{R}+,\ \mathrm{R}+) \tag{109}$$

where the signs in the various sectors refer to the values respectively taken by $(-1)^F$ and $(-1)^{\tilde{F}}$. In the (NS $-$, NS $-$) sector the lowest state is a tachyon, while the massless states live in the (NS $+$, NS $+$) sector. In the picture $(-1,-1)$ they are described by:

$$\psi^{\mu}_{-\frac{1}{2}}\tilde{\psi}^{\nu}_{-\frac{1}{2}}\ |0,\,\tilde{0},\,k\rangle_{(-1,-1)} \tag{110}$$

and are the same as in type II theories, namely a graviton, a dilaton and a Kalb-Ramond field. In the R-R sector, instead, we have the following massless states in the picture $(-\frac{1}{2},-\frac{1}{2})$:

$$u_A(k)\tilde{u}_B(k)|A\rangle_{-\frac{1}{2}}|\widetilde{B}\rangle_{-\frac{1}{2}}\ \ . \tag{111}$$

Since the terms containing G and $\tilde{G}$ in the second equation in (107) act as the identity on the previous state, the projector $P_{\rm R-R}$ imposes the existence of two kinds of R-R $(p+1)$-potentials for any value of p (C_{p+1} and $\bar{C}_{p+1}$) characterized respectively by:

$$u_A \left(\frac{1+\Gamma^{11}}{2}\right)^A_{\ B} = 0 \quad , \quad \tilde{u}_A \left(\frac{1\pm\Gamma^{11}}{2}\right)^A_{\ B} = 0 \tag{112}$$

and by

$$u_A \left(\frac{1-\Gamma^{11}}{2}\right)^A_{\ B} = 0 \quad , \quad \tilde{u}_A \left(\frac{1\mp\Gamma^{11}}{2}\right)^A_{\ B} = 0 \quad , \tag{113}$$

where the upper [lower] sign corresponds to 0B [0A]. The doubling of the R-R potentials implies the existence of two kinds of branes that are charged with respect to both potentials. We follow the convention of denoting by p and p' respectively branes having equal or opposite charges with respect to the two $(p+1)$ R-R potentials. The p and p'-branes are called respectively electric and magnetic branes .[22] In the case of a D3 brane the two potentials C_4 and $\bar{C}_4$ have field strengths F_5 and $\bar{F}_5$ being the former self-dual, $F_5 = {}^*F_5$, and the latter antiself-dual, $\bar{F}_5 = -{}^*\bar{F}_5$, as follows from Eq.s (112) and (113). This means that, if we take the linear combinations:

$$(C_4)^{\pm} = \frac{1}{\sqrt{2}}(C_4 \pm \bar{C}_4) \quad , \tag{114}$$

one can see that the Hodge duality transforms each field strength into the other according to the relation ${}^*F_5^{\pm} = F_5^{\mp}$. Therefore, while the D3 brane of type IIB is naturally dyonic, in type 0B the dyonic D3 brane is constructed as a superposition of an equal number of electric and magnetic D3 branes.[24, 71, 72]

Type 0 string theory can also be thought as the orbifold type IIB$/(-1)^{F_s}$, [74] where F_s is the space-time fermion number operator. From this point of view, the spectrum of the physical closed string states, written in Eq.s (108) and (109), is made of the untwisted and twisted sectors of this orbifold. Of course the untwisted spectrum coincides with the bosonic states of type II theories. The twisted sector can be more easily determined using the Green-Schwarz formalism, rather than the NS-R one, due to the simple action of $(-1)^{F_s}$ on the space-time fermionic coordinates S^{Aa}. Here $A = 1, 2$ and a labels the two spinor representations of the light-cone Lorentz group $SO(8)$, namely it is either an $\mathbf{8_s}$ or $\mathbf{8_c}$ index. In the twisted sector the boundary conditions on these coordinates are antiperiodic rather than periodic. Hence, the Fourier expansion for them contains half-integer fermionic modes. The lowest level corresponds to a tachyon while the first one, corresponding to the massless states, is given by:

$$S^a_{-\frac{1}{2}} \tilde{S}^b_{-\frac{1}{2}} |0\rangle| \otimes |\tilde{0}\rangle \qquad \text{in type 0B} \tag{115}$$

$$S^a_{-\frac{1}{2}} \tilde{S}^{\dot{b}}_{-\frac{1}{2}} |0\rangle \otimes |\tilde{0}\rangle \qquad \text{in type 0A.} \tag{116}$$

These states provide the doubling of the R-R forms previously discussed.

5.2 Boundary state

The presence of R-R potentials implies that type 0 theories contain D-branes that, as in other string theories, admit a microscopic description in the closed string channel in terms of boundary states. Obviously, the boundary state describing a 0B (or 0A) brane must be invariant under the GSO projectors defined in Eq. (107).

In type II theories the boundary state is constructed in terms of the state $|B, \eta\rangle$ with $\eta = \pm 1$ by imposing the standard GSO projection which selects the following invariant combinations:

$$|B\rangle = \frac{1}{2}\Big(|B, +\rangle_{\rm NS-NS} - |B, -\rangle_{\rm NS-NS} + |B, +\rangle_{\rm R-R} + |B, -\rangle_{\rm R-R}\Big) \quad . \tag{117}$$

In type 0 theories it is simple to verify that the boundary state $|B, \eta\rangle$ that one uses, independently on the spin structures, is already invariant under the GSO operators (107). This means that in type 0 string we have, for each p, four different kinds of boundary states:

$$|Bp, \eta, \eta'\rangle = |Bp, \eta\rangle_{\rm NS-NS} + |Bp, \eta'\rangle_{R-R} \quad . \tag{118}$$

However the requirement of consistency of the various cylinder amplitudes, giving in the closed channel the interaction between electric and magnetic branes, with the corresponding amplitudes computed in the open string channel, imposes the following combinations:[76, 77]

$$|Bp, \pm\rangle = \pm|Bp, \pm\rangle_{\rm NS-NS} + |Bp, \pm\rangle_{R-R} \quad , \tag{119}$$

which are indeed the boundary state descriptions of the p and p' branes (respectively for $\eta = +, -$) already introduced, as one can check by evaluating the coupling of the previous boundary states with the two forms C_{p+1} and $\bar{C}_{p+1}$.

Notice that, because of the $\pm$ sign in front of the NS-NS term in Eq. (119), electric and magnetic branes have opposite couplings with the tachyon, which implies that dyonic branes do not couple to it.

The normalization coefficient of the boundary state (related to the tension of the Dp brane [10]) in type 0 string is:[22]

$$T_p = \frac{T_p^{II}}{\sqrt{2}} \quad . \tag{120}$$

An easy way to see this is the following: in the open channel the interaction between two Dp branes of the same type coincides with the corresponding expression in the NS-sector of type II theories. In the closed string channel, instead, in the expression for the type 0 a factor $1/2$ is missing with respect to the one in type II, due to the different GSO projection, and therefore the tension of the branes must be $1/\sqrt{2}$ smaller.

[10] See Ref. [57] for details.

5.3 Open string spectrum

The existence of two different kinds of branes in type 0 theories (the p-brane and the p'-brane) implies the presence of four distinct kinds of open strings: those stretching between two p or two p'-branes (denoted by pp and $p'p'$) and those of mixed type (pp' and $p'p$). This means that the most general Chan-Paton factor λ in the expression of the open string states has the following form:

$$\lambda \equiv \begin{pmatrix} pp & pp' \\ p'p & p'p' \end{pmatrix} . \tag{121}$$

Open/closed string duality makes the following spin structure correspondence to hold:[22]

$$\begin{array}{lll} \text{Interactions} & \text{Closed states} & \text{Open states} \\ pp \;\; p'p' & \mathrm{NS}-\mathrm{NS} & \mathrm{NS} \\ pp \;\; p'p' & \mathrm{R}-\mathrm{R} & \mathrm{NS}(-1)^F \\ pp' \;\; p'p & \mathrm{NS}-\mathrm{NS}(-1)^F & \mathrm{R} \\ pp' \;\; p'p & \mathrm{R}-\mathrm{R}(-1)^F & \mathrm{R}(-1)^F \end{array} \tag{122}$$

From this scheme, it is easy to see that the spectrum of pp and $p'p'$ strings contains only the NS and $\mathrm{NS}(-1)^F$ sectors [22] whose massless excitations are the bosons of the gauge theory. In the case of $D3$ branes one has:

$$A^\alpha \equiv \begin{pmatrix} pp & 0 \\ 0 & p'p' \end{pmatrix} \otimes \psi^\alpha_{-1/2}|0\rangle_{-1} \qquad \alpha = 0, \ldots, 3 \tag{123}$$

$$\phi^i \equiv \begin{pmatrix} pp & 0 \\ 0 & p'p' \end{pmatrix} \otimes \psi^i_{-1/2}|0\rangle_{-1} \qquad i = 4, \ldots, 9 \;\; . \tag{124}$$

Here A^α corresponds to the gauge field, while the ϕ^i's represent six adjoint scalars. On the other hand pp' strings have only the R spectrum [22] which provides fermions to the gauge theory supported by the branes. The lowest excitations of these strings are:

$$\psi^A \equiv \begin{pmatrix} 0 & pp' \\ p'p & 0 \end{pmatrix} \otimes |A\rangle_{-\frac{1}{2}} \;\; . \tag{125}$$

being $|A\rangle_{-\frac{1}{2}}$ a Majorana-Weyl spinor of the ten-dimensional Lorentz group.

Finally, on a dyonic brane there are both fermionic and bosonic degrees of freedom. Indeed a stack of N dyonic D-branes of type 0 contains N p-branes and N p'-branes, and therefore the massless open string states living on their world-volume are the subset of the massless open string states living on the world-volume of $2N$ Dp branes of type II theories, that are invariant under the action of the operator $(-1)^{F_s}$, where we are looking at type 0B as the orbifold IIB$/(-1)^{F_s}$. In order to select the invariant open string states, one has to pay attention to the action of the space-time fermion number operator on the Chan-Paton factors.[72, 33] Indeed consistency requirements between the two approaches

to type 0B theory impose the following non trivial action of $(-1)^{F_s}$ on the Chan-Paton factors:

$$(-1)^{F_s}\lambda_{ij} \equiv \left(\gamma_{(-1)^{F_s}}\right)_{ih} \lambda_{hk} \left(\gamma^{-1}_{(-1)^{F_s}}\right)_{kj} \quad , \tag{126}$$

where [33]

$$\gamma_{(-1)^{F_s}} = \begin{pmatrix} \mathbb{I}_{N\times N} & 0 \\ 0 & -\mathbb{I}_{M\times M} \end{pmatrix} \tag{127}$$

and N, M denote the number of p and p'-branes respectively. The requirement of invariance of the physical states under the action of $(-1)^{F_s}$ imposes the following constraints on the Chan-Paton factors:

$$\lambda^{(\mathrm{NS})} = \gamma_{(-1)^{F_s}}\lambda^{(\mathrm{NS})}\gamma^{-1}_{(-1)^{F_s}} \qquad \lambda^{(\mathrm{R})} = -\gamma_{(-1)^{F_s}}\lambda^{(\mathrm{R})}\gamma^{-1}_{(-1)^{F_s}}, \tag{128}$$

where the minus sign is due to the action of F_s on the space-time fermion $|A\rangle$. It is easy to see that the previous equations are satisfied by the matrices given in Eqs. (123), (124) and (125).

Therefore the spectrum of the open strings attached on a dyonic brane can be easily derived by writing, in the NS and R sectors, the massless states:

$$A_\alpha \equiv \begin{pmatrix} A_{NN} & 0 \\ 0 & B_{NN} \end{pmatrix} \psi^\alpha_{-1/2}|0,k\rangle \qquad \alpha = 0,\ldots,3 \tag{129}$$

$$\phi^i \equiv \begin{pmatrix} A_{NN} & 0 \\ 0 & B_{NN} \end{pmatrix} \psi^i_{-1/2}|0,k\rangle \qquad i = 4\ldots 9 \tag{130}$$

$$\Psi^i \equiv \begin{pmatrix} 0 & A_{NN} \\ B_{NN} & 0 \end{pmatrix} |s_1\, s_2\, s_3\, s_4\rangle \qquad \sum_{i=1}^4 s_i = odd \quad , \tag{131}$$

where the last relation between the s_i follows from the GSO projection in Eq.s (36) and (39).

Thus the world-volume of a dyonic D3 brane configuration supports a $U(N)\times U(N)$ gauge theory with six adjoint scalars for each gauge factor and four Weyl fermions in the bifundamental representation of the gauge group $(N\,,\,\bar{N})$ and $(\bar{N}\,,\,N)$ (see Table 1).

The number of bosonic degrees of freedom of the open strings attached on the N dyonic branes is $8N^2\times 2$, and coincides with the number of the fermionic ones. Therefore, the gauge theory supported by these bound states, even if non-supersymmetric, exhibits a Bose-Fermi degeneracy. Its β-function is zero at one-loop level, and it is argued that this non-supersymmetric theory is conformal in the large N limit.[24, 75]

5.4 One-loop vacuum amplitude

In this section we compute the interaction between two branes in type 0B theory, first in the absence of an external field and then turning on an $SU(N)$ gauge field on one of

them. As usual, this interaction can be computed either in the open string channel or in the closed string one.

In the open channel the interaction between two branes of the same kind is:[22]

$$
\begin{aligned}
Z^o_{pp} &= 2\int \frac{d\tau}{2\tau} \mathrm{Tr}_{\mathrm{NS}} \left[e^{-2\pi\tau L_0} (-1)^{G_{bc}} P_{GSO} \right] \\
&= V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{1}{2} \left[\left(\frac{f_3(k)}{f_1(k)} \right)^8 - \left(\frac{f_4(k)}{f_1(k)} \right)^8 \right] \quad , \qquad (132)
\end{aligned}
$$

with P_{GSO} defined in Eq. (36). The interaction between a p and a p'-brane is obtained by computing the trace in the R-sector:[22]

$$
\begin{aligned}
Z^o_{pp'} &= 2\int \frac{d\tau}{2\tau} \mathrm{Tr}_{\mathrm{R}} \left[e^{-2\pi\tau L_0} (-1)^{G_{bc}} P_{GSO} \right] \\
&= -V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{1}{2} \left(\frac{f_2(k)}{f_1(k)} \right)^8 \quad . \qquad (133)
\end{aligned}
$$

Thinking of type 0B as type IIB$/(-1)^{F_s}$, Eq.s (132) and (133) can be written in a more compact form by introducing, in the trace of the free energy, the following projector:

$$
P_{(-1)^{F_s}} = \frac{1+(-1)^{F_s}}{2} \qquad (134)
$$

that, as we have previously mentioned, eliminates all fermionic states from the spectrum in the closed channel. The free-energy is now written as:

$$
\begin{aligned}
& Z^o = 2\int \frac{d\tau}{2\tau} \mathrm{Tr}_{\mathrm{NS-R}} \left[e^{-2\pi\tau L_0} (-1)^{G_{bc}} P_{\mathrm{GSO}} P_{(-1)^{F_s}} \right] \\
& = \frac{1}{2} \mathrm{Tr}\left[\mathbb{I}\right]^2 V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{1}{2} \left[\left(\frac{f_3(k)}{f_1(k)} \right)^8 - \left(\frac{f_4(k)}{f_1(k)} \right)^8 - \left(\frac{f_2(k)}{f_1(k)} \right)^8 \right] \\
& + \frac{1}{2} \mathrm{Tr}\left[\gamma_{(-1)^{F_s}}\right]^2 V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \frac{1}{2} \left[\left(\frac{f_3(k)}{f_1(k)} \right)^8 - \left(\frac{f_4(k)}{f_1(k)} \right)^8 + \left(\frac{f_2(k)}{f_1(k)} \right)^8 \right] \\
& \equiv Z^o_{pp} + 2 Z^o_{pp'} + Z^o_{p'p'} \quad . \qquad (135)
\end{aligned}
$$

Since the traces of the Chan-Paton factors are given by

$$
\mathrm{Tr}\left[\mathbb{I}\right]^2 = (N+M)^2 \qquad \mathrm{Tr}\left[\gamma_{(-1)^{F_s}}\right]^2 = (N-M)^2, \qquad (136)
$$

we can rewrite the previous equation as follows:

$$
Z^o = V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}}
$$

$$
\times \left\{ \frac{N^2+M^2}{2} \left[\left(\frac{f_3(k)}{f_1(k)} \right)^8 - \left(\frac{f_4(k)}{f_1(k)} \right)^8 \right] - MN \left(\frac{f_2(k)}{f_1(k)} \right)^8 \right\} . \qquad (137)
$$

We see that Eq.s (132) and (133) are obtained by putting respectively $M=0$, $N=1$ and $N=M=1$ in Eq. (137), while by taking $N=M$ we get the interaction among N dyonic branes.

Following Ref. [17], whose results are outlined in Sect. 2, we compute now the one-loop vacuum amplitude of an open string stretching between a D3 brane dressed with an external $SU(N)$ gauge field and a stack of N D3 branes, located at a distance y from the first one.

In the open channel the interaction between two branes *of the same kind* can be read from Eq. (40) by taking only the contribution of the spin structures NS and NS $(-1)^F$ and multiplying it by a factor 2 (there is no orbifold projection in this case), while the closed channel expression can be obtained from Eq. (45) by considering only the spin structures NS-NS and R-R always multiplied by the same factor. From Eq. (40) it follows that the contributions to the term F^2 coming from the NS and NS$(-1)^F$ spin structures exactly cancel the one coming from the R spin structure. Therefore, instead of considering the sum of the spin structures NS and NS $(-1)^F$ in Eq. (40), we can consider only the spin structure R, which is equal to the first two, up to a sign. By using Eq.s (391), (392) and (394) of Appendix B and inserting them in the last term in Eq. (40), after having changed its sign and multiplied it by a factor 2, the coefficient of the gauge kinetic term turns out to be:

$$\frac{1}{g^2_{YM}} = -\frac{2N}{16\pi^2}\int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \left(\frac{f_2(k)}{f_1(k)}\right)^8 \left[\frac{1}{12\tau^2} + k\frac{d}{dk}\log f_2(k)\right] . \tag{138}$$

Differently from the type IIB case, in type 0B, there are threshold corrections to the running coupling constant. In order to select only the contribution of the massless states and to compare it with the gauge theory expectation, we can perform the field theory limit obtaining:

$$\frac{1}{g^2_{YM}} = -\frac{2N}{16\pi^2}\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \times \frac{16}{12} = \frac{N}{(4\pi)^2}\frac{8}{3}\log\frac{y^2}{2\pi(\alpha')^2\Lambda^2} \quad , \tag{139}$$

that agrees with the expected behaviour of the running coupling constant of a gauge theory with one vector and six adjoint scalars!

In the closed string channel one can consider only the contribution of the spin structure NS-NS $(-1)^F$ in Eq. (45) which exactly cancels the ones of the NS-NS and R-R spin structures. Here, by making use of the expansions given in Eq.s (397), (398) and (394), and inserting them in the second term in Eq. (45) (after having changed its sign and multiplied by a factor 2) we get the following expression for the running coupling constant in the closed channel:

$$\frac{1}{g^2_{YM}} = \frac{2N}{16\pi^2}\int_0^\infty \frac{dt}{t^3} e^{-\frac{y^2}{2\pi\alpha' t}} \left(\frac{f_4(q)}{f_1(q)}\right)^8 \left[-\frac{1}{12} + q\frac{d}{dq}\log f_4(q)\right] \quad . \tag{140}$$

By using Eq. (101) and the modular properties of f_1 and f_2 (see Eq. (364)) it can be seen that Eq. (140) reduces to Eq. (138). However, by performing the field theory limit ($\alpha' \to 0$ and $t \to \infty$ with $\alpha' t$ finite) in the closed channel, one selects the massless states contribution to the one loop running of the coupling constat:

$$\frac{1}{g^2_{YM}} = -\frac{N}{64\pi^2}\int_0^\infty \frac{dt}{t^3} e^{-\frac{y^2}{2\pi\alpha' t}} \left(e^{\pi t} - 8\right) \quad . \tag{141}$$

In this limit the closed string tachyon gives a divergent contribution, while the massless states give a vanishing one. In other words, the field theory limit of Eq. (140) does not reproduce the correct answer for the running coupling constant, revealing that the gauge/gravity correspondence does not work in this case.

In this section we have computed the one-loop vacuum amplitude of an open string stretching between a stack of N D3 branes and another brane of the same kind dressed with an $SU(N)$ gauge field. An important question is, however, whether the stack of N D-branes would fly apart. In order to understand this point one should look more carefully at the interaction between two identical branes in absence of any background field. Performing the modular transformation $\tau \to 1/t$ on Eq. (132) and extracting the contribution of each closed string mass level, it turns out that for those levels corresponding to an even power of q the R-R repulsion is twice the NS-NS attraction, while for those corresponding to odd powers of q only the NS-NS states contribute with an attractive term. Hence, level by level, the contribution to the interaction is always different from zero. One can conclude that, unless some geometrical constraint arises to force the system to be stable, there are problems in piling up N identical branes on top of each other. The authors of Ref. [22, 78] argue that a stable bound state is formed when the tachyon mass is shifted to remove the instability. In this review we will not try to make this argument more quantitative, because little is known about the tachyon condensation in closed string. However this problem can be avoided by considering a dyonic configuration. From Eq. (137) one easily deduces that the zero-force condition between two stacks made of a superposition of an equal number of electric and magnetic D3 branes is indeed satisfied. In terms of open string states, the interaction between two dyonic branes vanishes because of a cancellation between the contribution of bosonic and fermionic degrees of freedom, ensuring the stability of the configuration. On the same footing, the one-loop vacuum amplitude of an open string stretching between a stack of N dyonic branes and a further dyonic brane dressed with an $SU(N)$ gauge field, is twice the corresponding one in type IIB theory, and in particular, the coefficient of the gauge kinetic term turns out to be identically zero, leading to the correct vanishing of the one-loop beta function, both in the open and closed channel. Hence for a dyonic brane, the gauge/gravity correspondence holds.

One can also consider orbifolds of type 0B string theory and, by introducing dyonic fractional branes which live at orbifold singularities, one breaks the conformal invariance of the model, obtaining a non-supersymmetric and non-conformal gauge theory. These are examples of the so-called *orbifold field theories* [11], which are non-supersymmetric gauge theories (daughter theories) that in the planar limit are perturbatively equivalent to some supersymmetric gauge theories (parent theories). The gauge groups of the daughter and parent theories are not the same: in the present case the gauge theory living on N dyonic branes has an $SU(N)\times SU(N)$ gauge symmetry [12], while the corresponding parent theory

[11] See Ref. [34] and Ref.s therein. See also Ref. [79].

[12] The factor $U(1)$ is neglected in the decomposition $U(N) = SU(N)\times U(1)$.

is an $SU(2N)$ SYM, and also the field content is actually different. However there exists a common sector in the spectrum for which scattering amplitudes in the parent and daughter theory are the same in the planar limit.[25, 26] The world-volume gauge theories living on N dyonic D3 branes in flat space and its orbifolds $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ are listed in Table 1.

Table 1: Spectrum of the $SU(N)$ world-volume gauge theory living on N dyonic D3 branes.

	0B *on flat space*	0B *on* $\mathbb{C}^2/\mathbb{Z}_2$	0B *on* $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$
Gauge vector	(Adj,1)+(1,Adj)	(Adj,1)+(1,Adj)	(Adj,1)+(1,Adj)
Scalars	6[(Adj,1)+(1,Adj)]	2[(Adj,1)+(1,Adj)]	–
Weyl fermions	$4[(\Box,\bar{\Box})+(\bar{\Box},\Box)]$	$2[(\Box,\bar{\Box})+(\bar{\Box},\Box)]$	$(\Box,\bar{\Box})+(\bar{\Box},\Box)$
1-loop β-function	0	$-\frac{N}{8\pi^2}g_{YM}^3$	$-\frac{3N}{16\pi^2}g_{YM}^3$
	($\mathcal{N}=4$ SYM)	($\mathcal{N}=2$ SYM)	($\mathcal{N}=1$ SYM)

Let us first analyse the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. This orbifold, in the case of fractional branes, acts only on the oscillator part of the states in Eq.s (129), (130) and (131). The action on the bosonic and fermionic coordinates is given in Eq. (17), while that on the R-R vacuum is given by:

$$h : |s_1\, s_2\, s_3\, s_4\rangle \longrightarrow e^{i\pi(s_3+s_4)}|s_1\, s_2\, s_3\, s_4\rangle \quad . \tag{142}$$

The states left invariant are, for each gauge group, one gauge vector, two scalars in the adjoint, two Weyl fermions in the $(N\,,\bar{N})$ representation and two Weyl fermions in the $(\bar{N}\,,N)$ one. It is simple to check that the β-function of the daughter theory relative to each gauge group is, at one-loop level, the same as the pure $\mathcal{N}=2$ super Yang-Mills with gauge group $SU(2N)$, if the gauge coupling of the latter g_P is related to the one of the former g_D by $g_D^2=2g_P^2$.[34]

Analogously one can analyse the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ acting on the coordinates according to Eq. (79) and on the fermions as follows:

$$\begin{aligned} h_1|s_1\, s_2\, s_3\, s_4\rangle &= e^{i\pi(s_3+s_4)}|s_1\, s_2\, s_3\, s_4\rangle \\ h_2|s_1\, s_2\, s_3\, s_4\rangle &= e^{i\pi(s_2+s_4)}|s_1\, s_2\, s_3\, s_4\rangle \\ h_3|s_1\, s_2\, s_4\, s_4\rangle &= -e^{i\pi(s_2+s_3)}|s_1\, s_2\, s_3\, s_4\rangle \quad , \end{aligned} \tag{143}$$

where the sign in front of the last equation is required by the group properties. The states left invariant are, for each gauge group, one gauge vector, one Weyl fermion in $(N\,,\bar{N})$ and one Weyl fermion in $(\bar{N}\,,N)$. It is simple to check that the β-function of each gauge factor of the daughter theory is, at one-loop, the one of $\mathcal{N}=1$ super Yang-Mills with gauge group $SU(2N)$, with the appropriate rescaling of the running coupling constant. Notice that for $N=3$, if one of the two $SU(N)$ factors is interpreted as a colour index and the

other as a flavour index, this model provides an example of a three-colour/three-flavour QCD .[34]

The interesting aspect of these orbifold theories is that the expression of the one-loop vacuum amplitude of an open string stretching between a stack of N dyonic branes and one electric or magnetic brane dressed with an $SU(N)$ gauge field coincides with the corresponding one computed in type IIB and studied in Ref. [17]. This follows from the fact that, because of Eq.s (119) and (120), the boundary state of N dyonic branes is the same as the one of type IIB and therefore, when we multiply it by a closed string propagator and sandwich it with the boundary state of an electric or magnetic brane, we get exactly the same result as in type IIB theory. Therefore, all the features discussed in Sect. 3 are also shared by these non-supersymmetric theories. In particular the gauge/theory parameters do not admit threshold corrections either in the open or in the closed channel and the gauge/gravity correspondence perfectly holds.

6 Gauge/Gravity Correspondence in Type $0'$ Theories

In this section we are going to study the gauge/gravity correspondence for type $0'$ theories.[29] These are unoriented, non-supersymmetric string models that can be constructed as orientifolds of type 0 theories by taking the quotient 0B $/\Omega'$, where Ω' is a suitable world-sheet parity operator which imposes the existence of a non trivial background made of an orientifold 9-plane (O9) and 32 D9 branes necessary to ensure the R-R tadpole cancellation, as we will discuss in detail later.

6.1 World-sheet parity definitions

In the literature one can find two different definitions of the world-sheet parity operator Ω. In particular, the following definition is given in the closed string sector in Ref.s [80] and [81]:

$$\begin{aligned}
\tilde{\Omega}\alpha_n^\mu\tilde{\Omega}^{-1} &= \tilde{\alpha}_n^\mu \\
\tilde{\Omega}\psi_r^\mu\tilde{\Omega}^{-1} &= \tilde{\psi}_r^\mu \\
\tilde{\Omega}\tilde{\psi}_r^\mu\tilde{\Omega}^{-1} &= -\psi_r^\mu
\end{aligned} \tag{144}$$

and

$$\begin{aligned}
\tilde{\Omega}\left(|0\rangle_{-1}\otimes|\tilde{0}\rangle_{-1}\right) &= |0\rangle_{-1}\otimes|\tilde{0}\rangle_{-1} \\
\tilde{\Omega}\left(|A\rangle_{-1/2}\otimes|\tilde{B}\rangle_{-1/2}\right) &= -|B\rangle_{-1/2}\otimes|\tilde{A}\rangle_{-1/2}\ ,
\end{aligned} \tag{145}$$

where the NS-NS and the R-R vacua are taken respectively in the $(-1,-1)$ and $(-\frac{1}{2},-\frac{1}{2})$ symmetric pictures. Notice that the operator $\tilde{\Omega}$ satisfies the condition $\tilde{\Omega}^2 = (-1)^{F+\tilde{F}}$, where $(-1)^F$ is defined as in Eq.s (38) and (39) with the analogous definition for $(-1)^{\tilde{F}}$. In the open string sector the world-sheet parity operator $\tilde{\Omega}$ acts as follows:

$$\tilde{\Omega}\alpha_m\tilde{\Omega}^{-1} = \pm e^{i\pi m}\alpha_m \qquad \tilde{\Omega}\psi_r\tilde{\Omega}^{-1} = \pm e^{i\pi r}\psi_r \tag{146}$$

for integer and half-integer r and the signs $\pm$ refer respectively to NN and DD boundary conditions.

From Eq.s (3.11) and (3.12) of Ref. [80] it is possible to get the action of $\tilde{\Omega}$ on the NS vacuum:

$$\tilde{\Omega}\,|0\rangle_{-1} = -i\,|0\rangle_{-1} \tag{147}$$

and, in order to get a supersymmetric theory, the following action must hold on the R vacuum:

$$\tilde{\Omega}|A\rangle_{-1/2} = -\left(\Gamma^{p+1}\dots\Gamma^{9}\right)^{A}{}_{B}\,|B\rangle_{-1/2} \quad , \tag{148}$$

where we have NN boundary conditions along the world-volume of Dp brane corresponding to the directions $0, 1, \dots, p$ and DD boundary conditions along the remaining ones. In Ref.s [82], [21] and [83] a different definition of the world-sheet parity operator Ω is used.

In the closed string channel one has:

$$\begin{aligned} \Omega\alpha_n^\mu\Omega^{-1} &= \tilde{\alpha}_n^\mu \\ \Omega\psi_r^\mu\Omega^{-1} &= \tilde{\psi}_r^\mu \\ \Omega\tilde{\psi}_r^\mu\Omega^{-1} &= \psi_r^\mu \end{aligned} \tag{149}$$

with $\Omega^2 = 1$ and

$$\begin{aligned} \Omega\left(|0\rangle_{-1}\otimes|\tilde{0}\rangle_{-1}\right) &= -|0\rangle_{-1}\otimes|\tilde{0}\rangle_{-1} \\ \Omega\left(|A\rangle_{-1/2}\otimes|\tilde{B}\rangle_{-1/2}\right) &= -|B\rangle_{-1/2}\otimes|\tilde{A}\rangle_{-1/2} \quad , \end{aligned} \tag{150}$$

while in the open string channel the definition is like the corresponding one given by $\tilde{\Omega}$. It is straightforward to see that for type IIB theory, the two definitions are equivalent and both of them yield type I theory, i.e.:

$$\text{type I} = \text{ type IIB}/\Omega = \text{ type IIB}/\tilde{\Omega} \quad . \tag{151}$$

It is important here to observe that one can define the world-sheet parity action on the closed string R-R vacuum, given in Eq.s (145) and (150), only in type II/0B theories where the GSO projection imposes, in the symmetric picture, that the spinors have the same chirality in both left and right sectors. As a consequence, Eq.s (145) and (150) define the world-sheet parity action only on the following combination of states:

$$\tilde{\Omega}\,\left(|\alpha\rangle\otimes|\tilde{\beta}\rangle\,,\,|\dot{\alpha}\rangle\otimes|\dot{\tilde{\beta}}\rangle\right) = -\left(|\beta\rangle\otimes|\tilde{\alpha}\rangle\,,\,|\dot{\beta}\rangle\otimes|\dot{\tilde{\alpha}}\rangle\right) \quad , \tag{152}$$

with an analogous expression for Ω, after having used the Γ-matrices in the chiral base, given in Ref.s [84] and [85], to decompose the 32-dimensional spinor $|A\rangle$ into two 16-dimensional spinors with opposite chirality: $|A\rangle \equiv |\alpha, \dot{\alpha}\rangle$, This remark will be crucial when we analyze the world-sheet parity action on the boundary state. The boundary state, in the R-R sector, is naturally written in the asymmetric picture $(-1/2\,,\,-3/2)$,

in which the GSO projected states of the R-R sector are constructed starting from 16-dimensional spinors having opposite chirality.[86] The world-sheet parity action on these spinors, for the reasons explained before, is not well-defined and, as we will see next, its definition has to be given in a consistent way.

When used to construct an orientifold of type 0 theory, Ω and $\tilde{\Omega}$ do not yield any interesting theory. Indeed the former gives a tachyon-free theory having the same kinds of branes as type I (hence there are no D3 branes) while the latter gives a theory with tachyons. An alternative definition of the world-sheet parity operator allows to get a tachyon-free orientifold theory of type 0B having D3 branes in the spectrum. It is given by the following operator:

$$\Omega' = \tilde{\Omega}(-1)^{\tilde{F}} \quad \text{with} \quad \Omega'^2 = 1 \quad . \tag{153}$$

The action of Ω' in the closed string sector is:

$$\begin{aligned} \Omega' \alpha_n^\mu \Omega'^{-1} &= \tilde{\alpha}_n^\mu \\ \Omega' \psi_r^\mu \Omega'^{-1} &= \tilde{\psi}_r^\mu \\ \Omega' \tilde{\psi}_r^\mu \Omega'^{-1} &= \psi_r^\mu \end{aligned} \tag{154}$$

and

$$\Omega' \left(|0\rangle_{-1} \otimes |\tilde{0}\rangle_{-1} \right) = -|0\rangle_{-1} \otimes |\tilde{0}\rangle_{-1} \tag{155}$$

$$\Omega' \left(|A\rangle_{-1/2} \otimes |\tilde{B}\rangle_{-1/2} \right) = \left(\Gamma^{11} \right)^B{}_D \, |D\rangle_{-1/2} \otimes |\tilde{A}\rangle_{-1/2} \quad , \tag{156}$$

where in the last equality we have used the following identity:

$$(-1)^{\tilde{F}} \left(|A\rangle_{-1/2} \otimes |\tilde{B}\rangle_{-1/2} \right) = - \left(\Gamma^{11} \right)^B{}_D \, |A\rangle_{-1/2} \otimes |\tilde{D}\rangle_{-1/2} \quad . \tag{157}$$

In particular, for the same reasons explained soon after Eq. (151), Eq. (156) defines the world-sheet parity action only on the spinors having the same chirality in both sectors. This means that, introducing the 16-dimensional basis for the left and right spinors, we have:

$$\Omega' \left(|\alpha\rangle \otimes |\tilde{\beta}\rangle \, , \, |\dot{\alpha}\rangle \otimes |\dot{\tilde{\beta}}\rangle \right) = \left(|\beta\rangle \otimes |\tilde{\alpha}\rangle \, , \, -|\dot{\beta}\rangle \otimes |\dot{\tilde{\alpha}}\rangle \right) \quad . \tag{158}$$

It can be seen from the definition of Ω' that it acts, in the open string channel, like both Ω and $\tilde{\Omega}$ do.

Notice that, being the action of Ω', Ω and $\tilde{\Omega}$ on physical states identical in type IIB theory, one can also define type I as:

$$\text{type I} = \text{type IIB}/\Omega' \quad . \tag{159}$$

6.2 Closed string spectrum

The spectrum of the type $0'$ theory is obtained from that of the type 0B one by selecting the Ω' invariant states.

Let us start from the closed string spectrum. In the NS-NS sector, from Eq. (155) it follows that the closed string tachyon is projected out, while at the massless level only the graviton and dilaton fields belong to the spectrum, being the Kalb-Ramond field projected out. For what concerns the R-R spectrum, the action of Ω' on the state in Eq. (111) is obtained by using Eq. (156). One gets:

$$\Omega' \left[(u_A(k)\tilde{u}_B(k)|A\rangle|\widetilde{B}\rangle) \right] = -(\pm)\tilde{u}_A(k)u_B(k)|A\rangle|\widetilde{B}\rangle \quad , \tag{160}$$

where the upper [lower] sign is valid for the upper sign in Eq. (112) [(113)] which corresponds to type 0B theory. By expanding in terms of a complete system of Γ-matrices, one can see that the R-R field is invariant under Ω' if the following condition is satisfied:

$$u_A(\Gamma_{\mu_1\ldots\mu_n}C^{-1})^{AB}\tilde{u}_B = -(\pm)u_A[(\Gamma_{\mu_1\ldots\mu_n}C^{-1})^T]^{AB}\tilde{u}_B \quad , \tag{161}$$

where the upper index T indicates the transposed matrix. But since one has, for odd n:

$$(\Gamma_{\mu_1\ldots\mu_n}C^{-1})^T = (-1)^{n(n-1)/2}(\Gamma_{\mu_1\ldots\mu_n}C^{-1}) \quad , \tag{162}$$

then Eq. (161) implies

$$1 = -(\pm)(-1)^{n(n-1)/2} \quad . \tag{163}$$

If we take the upper sign, the previous relation is satisfied for $n = 3$ corresponding to a R-R potential C_2, while if we take the lower sign it is satisfied for $n = 1, 5$ corresponding to the R-R potentials $\bar{C}_0$ and $\bar{C}_4$. This implies that the R-R massless bosonic spectrum of $0'$ theory is the same as IIB.

Moreover notice that because of Eq. (156), when acting on the original (R+, R+) sector of type 0B (where $\Gamma^{11} = 1$) Ω' leaves C_2 invariant and changes the sign of C_0 and C_4. On the contrary in the sector (R−, R−) (where $\Gamma^{11} = -1$) Ω' leaves $\bar{C}_0$ and $\bar{C}_4$ invariant and changes the sign of $\bar{C}_2$. In particular this implies that the action of Ω' on the combination $C_4^{\pm}$ defined in Eq. (114) is to send $C_4^{\pm} \to C_4^{\mp}$, up to a phase, namely it exchanges the role of the electric and magnetic branes. This suggests that the only Ω'-invariant Dp brane of type $0'$ would be a combination of the electric and magnetic branes of type 0B, as we are going to discuss in detail in the next subsection.

6.3 Boundary state

Let us now consider the brane content of type $0'$ theory, looking at the action of Ω' on the type 0 boundary states. The expression of the boundary state is given by:

$$|B\,,\eta\rangle = |B_X\rangle|B_\psi\,,\eta\rangle_{\mathrm{NS-NS,\,R-R}}|B_{\mathrm{gh.}}\rangle|B_{\mathrm{sgh.}}\,,\eta\rangle_{\mathrm{NS-NS,\,R-R}} \quad , \tag{164}$$

where the various factors in Eq. (164) can be found, for example, in Ref. [7]. In order to determine the world-sheet parity action on it, we first remind how such an operator acts on the ghost and superghost oscillators:

$$\Omega' b_n \Omega'^{-1} = \tilde{b}_n \qquad \Omega' \tilde{b}_n \Omega'^{-1} = b_n \qquad \Omega' c_n \Omega'^{-1} = \tilde{c}_n \qquad \Omega' \tilde{c}_n \Omega'^{-1} = c_n \tag{165}$$

$$\Omega' \beta_t \Omega'^{-1} = \tilde{\beta}_t \qquad \Omega' \tilde{\beta}_t \Omega'^{-1} = \beta_t \qquad \Omega' \gamma_t \Omega'^{-1} = \tilde{\gamma}_t \qquad \Omega' \tilde{\gamma}_t \Omega'^{-1} = \gamma_t \ , \tag{166}$$

while the action on the vacuum states, in the NS-NS sector, can be determined by observing that Ω', by definition, exchanges left and right sectors, i.e:

$$\Omega' \left[|0\rangle_{-1} \otimes |\tilde{0}\rangle_{-1} \right] = |\tilde{0}\rangle_{-1} \otimes |0\rangle_{-1} = |0\rangle_{-1} \otimes |\tilde{0}\rangle_{-1} \tag{167}$$

$$\Omega' \left[|q=1\rangle \otimes |\tilde{q}=1\rangle \right] = \tilde{c}_1 |\tilde{0}\rangle \otimes c_1 |0\rangle = |q=1\rangle \otimes |\tilde{q}=1\rangle \quad . \tag{168}$$

By using the previous transformations for the ghost and superghost degrees of freedom and Eq.s (154) and (155) we obtain:

$$\begin{aligned} \Omega' |B_X\rangle = |B_X\rangle \qquad & \Omega' |B_\psi, \eta\rangle_{\mathrm{NS-NS}} = -|B_\psi, -\eta\rangle_{\mathrm{NS-NS}} \\ \Omega' |B_{\mathrm{gh.}}\rangle = |B_{\mathrm{gh.}}\rangle \qquad & \Omega' |B_{\mathrm{sgh.}}\, \eta\rangle_{\mathrm{NS-NS}} = |B_{\mathrm{sgh.}}\, - \eta\rangle_{\mathrm{NS-NS}} \quad . \end{aligned} \tag{169}$$

These actions are compatible with the overlap conditions. For instance, the fermionic part of the boundary state satisfies the following overlap condition:

$$\Omega' \left(\psi_t^\mu - i\eta S^\mu_{\ \nu} \tilde{\psi}^\nu_{-t} \right) |B_\psi,\, \eta\rangle = -i\eta S^\mu_{\ \nu} \left(\psi^\nu_{-t} + i\eta S^\nu_{\ \rho} \tilde{\psi}^\rho_t \right) \Omega' |B_\psi,\, \eta\rangle = 0 \ , \tag{170}$$

where the index t is integer [half-integer] in the R-R [NS-NS] sector. This condition clearly shows that $\Omega'|B_\psi,\, \eta\rangle$ satisfies the same overlap as $|B_\psi,\, -\eta\rangle$ and therefore, apart from an overall factor, they can be identified.

At this point we are able to give the world-sheet parity action on the whole boundary state in the NS-NS sector:

$$\Omega' |Bp, \eta\rangle_{\mathrm{NS-NS}} = -|Bp, -\eta\rangle_{\mathrm{NS-NS}} \quad . \tag{171}$$

Like all physical states, also the boundary state must be invariant under the action of Ω'. The invariant boundary state has the following form:

$$|Bp\rangle_{\mathrm{NS-NS}} = \frac{1+\Omega'}{2} |Bp,\, +\rangle_{\mathrm{NS-NS}} = \frac{1}{2} \left[|Bp,\, +\rangle_{\mathrm{NS-NS}} - |Bp,\, -\rangle_{\mathrm{NS-NS}} \right] \ . \tag{172}$$

By comparing this equation with Eq. (119), one can easily see that, as far as the NS-NS sector is concerned, the boundary state describing a Dp brane in type $0'$ is the sum of the boundary states of the electric and magnetic branes of type 0B. This is consistent also with the comment at the end of the previous subsection.

Let us now consider the R-R sector where we have only to analyse the transformation properties of the fermionic and superghost part of the boundary states under the world-sheet parity. In particular, it is convenient to separate the zero-modes from the tower of oscillators (nzm), i.e.:

$$\begin{aligned} |B_\psi, \eta\rangle_{\mathrm{R-R}} &= |B_\psi, \eta\rangle^{(0)}_{\mathrm{R-R}} |B_\psi, \eta\rangle^{\mathrm{nzm}}_{\mathrm{R-R}} \\ |B_{\mathrm{sgh}}, \eta\rangle_{\mathrm{R-R}} &= |B_{\mathrm{sgh}}, \eta\rangle^{(0)}_{\mathrm{R-R}} |B_{\mathrm{sgh}}, \eta\rangle^{\mathrm{nzm}}_{\mathrm{R-R}} \quad . \end{aligned} \tag{173}$$

It is easy to verify that:

$$\Omega'|B_\psi, \eta\rangle^{\rm nzm}_{\rm R-R} = |B_\psi, -\eta\rangle^{\rm nzm}_{\rm R-R} \qquad \Omega'|B_{\rm sgh}, \eta\rangle^{\rm nzm}_{\rm R-R} = |B_{\rm sgh}, -\eta\rangle^{\rm nzm}_{\rm R-R} \quad . \tag{174}$$

What is less trivial is to determine the action of Ω' on the zero modes. Indeed the vacuum in the R-R sector is written in a picture which is not symmetric under the exchange of the right and left sectors. Furthermore, the action of Ω' must be defined in such a way to be compatible with the following definitions:

$$\begin{aligned} \psi_0^\mu |A\rangle \otimes |\tilde{B}\rangle &= \frac{1}{\sqrt{2}} \left(\Gamma^\mu\right)^A_{\ C} |C\rangle \otimes |\tilde{B}\rangle \\ \tilde{\psi}_0^\mu |A\rangle \otimes |\tilde{B}\rangle &= \frac{1}{\sqrt{2}} \left(\Gamma^{11}\right)^A_{\ C} \left(\Gamma^\mu\right)^B_{\ D} |C\rangle \otimes |\tilde{D}\rangle \quad , \end{aligned} \tag{175}$$

where $|A\rangle$ and $|\tilde{B}\rangle$ are 32-dimensional Majorana spinors.

It turns out that it is not possible to give a well-defined action of Ω' on such a kind of spinors that is compatible with Eq. (175). It is, however, possible to overcome this difficulty by giving the action of Ω' on the 16-dimensional chiral representation of the 32-dimensional spinors. In Eq. (158) this action is already given on the 16-dimensional spinors having the same chirality in the left and right sectors. Therefore, we have only to define as the world-sheet parity acts on spinors having opposite chiralities and it has to be defined in such a way to be compatible with Eq. (175). This requirement leads to:

$$\Omega' \begin{pmatrix} |\alpha\rangle \otimes |\tilde{\beta}\rangle & |\dot{\alpha}\rangle \otimes |\tilde{\beta}\rangle \\ |\alpha\rangle \otimes |\tilde{\dot{\beta}}\rangle & |\dot{\alpha}\rangle \otimes |\tilde{\dot{\beta}}\rangle \end{pmatrix} = \begin{pmatrix} |\beta\rangle \otimes |\tilde{\alpha}\rangle & |\beta\rangle \otimes |\tilde{\dot{\alpha}}\rangle \\ |\dot{\beta}\rangle \otimes |\tilde{\alpha}\rangle & -|\dot{\beta}\rangle \otimes |\tilde{\dot{\alpha}}\rangle \end{pmatrix} \quad . \tag{176}$$

Now we have all the ingredients to analyse how the zero-mode part of the boundary state transforms under Ω'. We remind that:[7]

$$|B_\psi,\, \eta\rangle^{(0)}_{\rm R-R} = \left[C\Gamma^0 \ldots \Gamma^p \left(1 + i\eta\Gamma^{11}\right)\right]_{AB} |A\rangle|\tilde{B}\rangle \quad , \tag{177}$$

where C is the charge conjugation matrix and the usual factor $(1+i\eta)^{-1}$ has been omitted here and will be included, for future convenience, in the superghost part of the boundary state.

Since p is odd then Eq. (177) contains only an even number of Γ-matrices that in the chiral basis are all anti-diagonal. This implies that the 32-dimensional spinors appearing in the vacuum of Eq. (177) can be decomposed in 16-dimensional chiral spinors, always having opposite chirality, i.e.

$$|A\rangle \otimes |\tilde{B}\rangle \equiv \Big(|\alpha\rangle \otimes |\tilde{\dot{\beta}}\rangle,\, |\dot{\alpha}\rangle \otimes |\tilde{\beta}\rangle\Big)\,. \tag{178}$$

In conclusion, for p odd, Ω' acts as follows on the 32-dimensional Majorana spinors:

$$\Omega'|A\rangle \otimes |\tilde{B}\rangle = |B\rangle \otimes |\tilde{A}\rangle \tag{179}$$

that implies the following action on the boundary state:

$$\Omega'|B_\psi,\, \eta\rangle^{(0)}_{\rm R-R} = (-1)^{\frac{5-p}{2}} |B_\psi,\, -\eta\rangle^{(0)}_{\rm R-R} \quad , \tag{180}$$

where we have used the identity

$$\left[C\Gamma^0 \dots \Gamma^p \left(1+\eta\Gamma^{11}\right)\right]^T = (-1)^{\frac{5-p}{2}} \left[C\Gamma^0 \dots \Gamma^p \left(1-\eta\Gamma^{11}\right)\right] \quad .$$

The same result can be obtained by following a different strategy. The zero modes of the R-R boundary states can be written in the following form:

$$|Bp,\eta\rangle^{(0)}_{\rm R-R} = 2^{\frac{p-9}{2}} \left(\psi^9_0 + i\eta\tilde{\psi}^9_0\right) \dots \left(\psi^{p+1}_0 + i\eta\tilde{\psi}^{p+1}_0\right) |B9,\eta\rangle^{(0)}_{\rm R-R} \ , \tag{181}$$

with

$$|B9,\eta\rangle^{(0)}_{\rm R-R} = \frac{1}{2^4} \left(\psi^0_0 - i\eta\tilde{\psi}^0_0\right) \dots \left(\psi^9_0 - i\eta\tilde{\psi}^9_0\right) C_{AB}|A\rangle \otimes |\tilde{B}\rangle \quad . \tag{182}$$

If we observe that

$$\Omega'|B9,\,\eta\rangle^{(0)}_{\rm R-R} = -(-i\eta)^{10}|B9,\,-\eta\rangle^{(0)}_{\rm R-R} = |B9,\,-\eta\rangle^{(0)}_{\rm R-R} \quad , \tag{183}$$

we can write:

$$\Omega'|Bp,\eta\rangle^{(0)}_{\rm R-R} = (i\eta)^{9-p}|Bp,-\eta\rangle^{(0)}_{\rm R-R} = -(-i\eta)^{7-p}|Bp,-\eta\rangle^{(0)}_{\rm R-R} \quad , \tag{184}$$

that coincides with Eq. (180) if p is odd.

Finally let us analyze the action of Ω' on the superghost boundary state. From the overlap conditions

$$(\gamma_t + i\eta\tilde{\gamma}_{-t})|B_{\rm sgh},\,\eta\rangle = 0 \qquad (\beta_t + i\eta\tilde{\beta}_{-t})|B_{\rm sgh},\,\eta\rangle = 0 \quad , \tag{185}$$

which hold for both NS-NS (t is half-integer) and R-R (t is an integer) sector, we deduce that $\Omega'|B_{sgh},\eta\rangle$ satisfies the same overlap condition as $|B_{sgh},-\eta\rangle$, therefore we conclude:

$$\Omega'|B_{sgh},\eta\rangle = k\,|B_{sgh},-\eta\rangle \qquad k^2 = 1 \quad , \tag{186}$$

together with

$$\Omega'|B_{sgh},\eta\rangle_{\rm R-R} = |B_{sgh},-\eta\rangle_{\rm R-R} \quad . \tag{187}$$

In order to find an expression of the boundary state $|B_{sgh},\,\eta\rangle_{\rm R-R}$ satisfying the conditions (185) for integer values of t, we notice that the overlap equations can be alternatively written in a different way by factorizing $i\eta$ and sending $t \to -t$.

$$(\tilde{\gamma}_t - i\eta\gamma_{-t})|B_{\rm sgh},\,\eta\rangle_{\rm R-R} = 0 \qquad (\tilde{\beta}_t - i\eta\beta_{-t})|B_{\rm sgh},\,\eta\rangle_{\rm R-R} = 0 \quad . \tag{188}$$

The boundary state $|B_{sgh},\,\eta\rangle_{\rm R-R}$ satisfying both conditions given in Eq.s (185) and (188) has to be symmetric under the exchange of γ with $\tilde{\gamma}$, β with $\tilde{\beta}$ and η with $-\eta$. The usual expression of the non-zero mode part of the boundary state, given for example in Ref. [7], trivially satisfies this latter symmetry. The zero-mode part, instead, has to be changed in order to be invariant under the world-sheet parity and it is natural to define it as follows:

$$|B_{\rm sgh},\eta\rangle^{(0)}_{\rm R-R} = \frac{1}{\sqrt{2}} \left[\frac{e^{i\eta\gamma_0\tilde{\beta}_0}}{1+i\eta} |0\rangle_{-\frac{1}{2}} \otimes |\tilde{0}\rangle_{-\frac{3}{2}} + \frac{e^{-i\eta\tilde{\gamma}_0\beta_0}}{1-i\eta} |\tilde{0}\rangle_{-\frac{1}{2}} \otimes |0\rangle_{-\frac{3}{2}} \right] , \tag{189}$$

where we have chosen for the phase factor the value $k = 1$ and, as previously said, we have included the factors $(1 \pm i\eta)^{-1}$. By using the well-known identity:

$$e^A\, e^B = e^B\, e^A\, e^{[A,\,B]} \tag{190}$$

valid when $[A,\, B]$ is a c-number and using the formulas:

$$\tilde{\gamma}_0 |\tilde{0}\rangle_{-\frac{3}{2}} = 0 \qquad \beta_0 |0\rangle_{-\frac{1}{2}} = 0 \tag{191}$$

one can get the following relations [13]:

$$\gamma_0 \exp[-i\eta\tilde{\gamma}_0\beta_0] = \exp[-i\eta\tilde{\gamma}_0\beta_0](\gamma_0 - i\eta\tilde{\gamma}_0) \qquad \tilde{\gamma}_0 \exp[-i\eta\tilde{\gamma}_0\beta_0] = \exp[-i\eta\tilde{\gamma}_0\beta_0]\tilde{\gamma}_0$$
$$\tilde{\beta}_0 \exp[-i\eta\tilde{\gamma}_0\beta_0] = \exp[-i\eta\tilde{\gamma}_0\beta_0](\tilde{\beta}_0 + i\eta\beta_0) \qquad \beta_0 \exp[-i\eta\tilde{\gamma}_0\beta_0] = \exp[-i\eta\tilde{\gamma}_0\beta_0]\beta_0 .$$

These equations allow us to prove that the boundary state in Eq. (189) satisfies the required overlap conditions. Finally, after collecting Eq.s (174), (180) and (187), we can write:

$$\Omega' |Bp,\, \eta\rangle_{\mathrm{R-R}} = (-1)^{\frac{9-p}{2}} |Bp,\, -\eta\rangle_{\mathrm{R-R}} \quad . \tag{192}$$

This means that the boundary state invariant under Ω' is the following:

$$|Bp\rangle_{\mathrm{R-R}} = \frac{1+\Omega'}{2} |Bp,\, +\rangle_{\mathrm{R-R}} = \frac{1}{2}\left[|Bp,\, +\rangle_{\mathrm{R-R}} + (-1)^{\frac{9-p}{2}} |Bp,\, -\rangle_{\mathrm{R-R}} \right] . \tag{193}$$

Hence in type $0'$ theory one has:

$$|Bp\rangle = |Bp\rangle_{\mathrm{NS-NS}} + |Bp\rangle_{\mathrm{R-R}} \quad , \tag{194}$$

with p odd, in agreement with the result of Ref. [30], where the first term is given in Eq. (172) and the second one in Eq. (193).

Eq.s (171) and (192) imply that the boundary states of type IIB theory, given in Eq. (117), is invariant under Ω' only for the values of $p = 1, 5, 9$. As a consequence, the only boundaries that we can have in type I theory are the ones associated with the D1, D5 and D9 branes. This remark can be seen as a check on the validity of our construction.

Let us notice that the boundary state given in Eq. (194) differs from the standard one given in the literature, only for the part regarding the superghost zero modes. However, it is simple to check that, in computing the interaction between branes, the contribution due to such modified boundary remains unchanged.

As previously mentioned in type $0'$ there is a non trivial background made of the O9-plane and a stack of 32 D9 branes. These latter are described by the boundary states with $p = 9$ that we have constructed in this subsection. For what concerns the O9-plane, it also admits a microscopic description in the language of perturbative closed string theory, which is given by the crosscap state that we explicitly construct in Subsect. 6.5.

[13] Remember that $[\gamma_0, \beta_0] = 1$.

6.4 Open string spectrum

In this subsection we determine the spectrum of the massless states living on the world-volume of N D3 branes of type $0'$ theory. We denote the generic open string state living in the world-volume of a Dp brane as:

$$|\chi,\, ij\rangle \equiv \lambda_{ij}|\chi\rangle \qquad i,j = 1\ldots N \quad , \tag{195}$$

where λ is an hermitian matrix [87] corresponding to the Chan-Paton factors that describe the gauge degrees of freedom and χ is the state made by the string oscillators. We denote by $\tilde{T}^{\tilde{a}}$, $\tilde{a} = 1,\ldots,N^2-1$ the generators of the group $SU(N)$ normalized as follows:

$$\mathrm{Tr}\left[\tilde{T}^{\tilde{a}}\,\tilde{T}^{\tilde{b}}\right] = \frac{1}{2}\delta^{\tilde{a}\tilde{b}} \quad . \tag{196}$$

By adding the properly normalized identity generator we obtain the generators of the group $U(N)$ given by $T^a \equiv \left(\tilde{T}^{\tilde{a}}\,,\frac{1}{\sqrt{2\,N}}\mathbb{1}\right)$ with $a = 1,\ldots,N^2$. They satisfy the relations:

$$\mathrm{Tr}\left[T^a T^b\right] = \frac{1}{2}\delta^{ab} \quad ; \quad \sum_{a=1}^{N^2} T^a_{ij}\,T^a_{kl} = \frac{1}{2}\delta_{il}\,\delta_{jk} \quad . \tag{197}$$

Since any hermitian matrix can be expanded in terms of the $U(N)$ generators we can write λ as follows:

$$\lambda_{ij}|\chi\rangle = \sum_{a=1}^{N^2} c_a (T^a)_{ij}|\chi\rangle \quad , \tag{198}$$

where the c_a's are arbitrary real coefficients. By considering:

$$\begin{aligned}\sum_{i,j=1}^{N}(T^b)_{ji}\lambda_{ij}|\chi\rangle &= \sum_{a=1}^{N^2} c_a (T^b)_{ji}(T^a)_{ij}|\chi\rangle = \\ &= \sum_{a=1}^{N^2} c_a \mathrm{Tr}[T^b\,T^a]|\chi\rangle = \frac{c_b}{2}|\chi\rangle \equiv |\chi, b\rangle\end{aligned} \tag{199}$$

and using Eq. (195), we can write:

$$|\chi, b\rangle = \sum_{i,j=1}^{N}(T^b)_{ji}|\chi, ij\rangle \qquad b = 1,\ldots,N^2 \quad . \tag{200}$$

Eq.s (195) and (200) give two different representations of the same open string state. In the following we are going to use the basis defined in Eq. (195). The two basis correspond to the two ways of representing a gauge field in field theory. Indeed we can represent it by a matrix $A^\mu_{ij} = \sum_{a=1}^{N^2} T^a_{ij} A^\mu_a$ or simply by A^μ_a. In particular, using the notation in Eq. (195), a massless gluon state of the open string can be written as follows:

$$c_a T^a_{ij}\epsilon_\mu \psi^\mu_{-1/2}|0\rangle \quad , \tag{201}$$

where c and ϵ are the two "polarization" vectors in the gauge and Minkowski space respectively.

As usual, physical states are left invariant by the world-sheet parity. In order to determine such states we also have to define the combined action of Ω' on the Chan-Paton factors and on the oscillators, given by:[80]

$$\Omega'|\chi,\, ij\rangle = \left(\gamma_{\Omega'_p}\right)_{im}|\Omega'\chi,\, nm\rangle\left(\gamma^{-1}_{\Omega'_p}\right)_{nj} \tag{202}$$

$$\Omega'^2|\chi,\, ij\rangle = \left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right)_{sj}\left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right)^{-1}_{ir}|\Omega'^2\chi,\, rs\rangle \quad . \tag{203}$$

The explicit form of the matrix $\gamma_{\Omega'_p}$ is obtained by imposing the constraint:

$$\Omega'^2|\chi,\, ij\rangle = |\chi,\, ij\rangle \quad , \tag{204}$$

with $\Omega'^2 = 1$ that is a consequence of the fact that physics is invariant under a double inversion of the string endpoints.

The massless states living on the world-volume of a D3 brane are given by:

$$\lambda_A\psi^{\alpha}_{-1/2}|0\rangle_{-1} \quad , \quad \lambda_\phi\psi^{i}_{-1/2}|0\rangle_{-1} \tag{205}$$

in the NS sector with $\alpha = 0, 1, 2, 3$ and $i = 4 \ldots 9$ and by

$$\lambda_\psi|A\rangle_{-1/2} \tag{206}$$

in the R sector. The generalization of the last two equations to a generic p brane is straightforward: one has just to notice that in this case $\alpha = 0, \ldots, p$ and $i = p+1, \ldots, 9$. In the following, by imposing the invariance under the action of Ω', we determine the precise structure of the Chan-Paton factors and show that it is the same as in Eq.s (123), (124) and (125).

In the NS sector it is easy to verify that $|\Omega'^2\chi\rangle = |\chi\rangle$ and therefore the Chan-Paton factors satisfy the condition:

$$\left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right)^{-1}\lambda_{A,\,\phi}\left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right) = \lambda_{A,\,\phi} \quad . \tag{207}$$

In the R-sector instead we have, for odd values of p:

$$|\Omega'^2\chi\rangle = \left(\Gamma^{p+1}\ldots\Gamma^9\right)^2|A\rangle = (-1)^{\frac{5-p}{2}}|A\rangle \tag{208}$$

and therefore the following condition has to be satisfied:

$$\left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right)^{-1}\lambda_\psi\left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right)(-1)^{\frac{5-p}{2}} = \lambda_\psi \quad . \tag{209}$$

The constraints given in Eq.s (207) and (209) seem to be uncompatible. But we have to remember that the matrix λ is diagonal in the NS sector (see Eq.s (123) and (124)) and off-diagonal in the R sector (see Eq. (125)) and therefore there is no contradiction.

In order to find an explicit expression of the world-sheet parity action on the Chan-Paton factors, satisfying both Eq.s (207) and (209), let us write, in the case of a single brane:

$$\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{210}$$

where of course $\det[\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}] = 1$ that implies $ad - bc = 1$ and therefore:

$$\left(\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right)^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad . \tag{211}$$

Eq. (207) imposes the following constraints:

$$b = 0 \qquad c = 0 \qquad a\,d = 1 \quad , \tag{212}$$

while Eq. (209) yields:

$$\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p} = \frac{1}{a}\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \qquad a^2 = (-1)^{\frac{5-p}{2}} \quad . \tag{213}$$

From the previous equation we see that $\gamma_{\Omega'_p}$ is symmetric or antisymmetric if $p = 1,5,9$ and that $\mathrm{Tr}[\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}] = 0$ for the D3 and D7 branes, in agreement with the results of Ref. [32]. By collecting the previous results we can write:

$$\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p} = \epsilon' e^{\epsilon i\pi\left(\frac{5-p}{4}\right)}\begin{pmatrix} (-1)^{\frac{5-p}{2}} & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{214}$$

where ϵ and ϵ' are equal to ± 1. In general one should consider ϵ and ϵ' dependent on p. However, as it will be shown in a while, the assumption that they are independent on p gives the right result when we compare the one-loop open string diagrams, which explicitly depend on the structure of the Chan-Paton factors, to the corresponding calculation in the closed string channel where instead we use the boundary state. The cancellation of the R-R tadpole in the case of the D9 brane implies that $\gamma_{\Omega'_9}$ is a symmetric matrix, [30] i.e. $\gamma^T_{\Omega'_9}\gamma^{-1}_{\Omega'_9} = +1$, which fixes $\epsilon' = -1$. This yields:

$$\mathrm{Tr}[\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}] = -2e^{i\pi\left(\frac{5-p}{4}\right)}\delta_{p,(1,5,9)} \quad . \tag{215}$$

Eq. (214) implies that the matrices $\gamma_{\Omega'_{9,1}}$ are symmetric, while the matrix $\gamma_{\Omega'_5}$ is antisymmetric. In order to find the explicit expression of $\gamma_{\Omega'_p}$ we have to impose Eq. (213). In so doing one gets:

$$\gamma_{\Omega'_p} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \qquad \text{with } \gamma = a\beta \quad . \tag{216}$$

However Eq. (213) does not fix either the determinant or the phase of $\gamma_{\Omega'_p}$. We can choose the determinant to be equal to (± 1) without changing the spectrum of the open strings

attached to a stack of branes of the same kind. For $p=3,7$, by choosing $\det\gamma_{\Omega'_p} = -\beta\gamma = -1$ and fixing the phase of $\gamma_{\Omega'_p}$ to be equal 1, one reproduces Eq. (71) of Ref. [32], i.e.:

$$\gamma_{\Omega'_p} = \begin{pmatrix} 0 & e^{\epsilon i\pi\left(\frac{5-p}{8}\right)} \\ e^{\epsilon i\pi\left(\frac{p-5}{8}\right)} & 0 \end{pmatrix} \qquad p=3,7 \quad . \tag{217}$$

By imposing the invariance of the states (205) under the action of Ω' and using Eq. (202) together with Eq.s (146) and (147), one finds the following constraints on the Chan-Paton factors:

$$\lambda_A = -\gamma_{\Omega'}\lambda_A^T\gamma_{\Omega'}^{-1} \qquad \lambda_\phi = \gamma_{\Omega'}\lambda_\phi^T\gamma_{\Omega'}^{-1} \quad , \tag{218}$$

that are satisfied by taking

$$\lambda_A = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \qquad \lambda_\phi = \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix}, \tag{219}$$

being A an $N\times N$ matrix. The last condition to be imposed on the Chan-Paton factors is $\lambda^\dagger = \lambda$ which fixes A to be an hermitian matrix, i.e. $A = A^\dagger$. The gauge group $U(N)$ acts on the Chan-Paton as follows:

$$\lambda \to \tilde{U}\lambda\tilde{U}^\dagger \quad , \tag{220}$$

where

$$\tilde{U} = \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}, \tag{221}$$

with U being an $N\times N$ matrix of the $U(N)$ group, and $\bar{U}$ being its complex conjugate. From Eq. (220) we get the following conditions:

$$A' = UAU^\dagger \qquad A'^T = \bar{U}A^T\bar{U}^\dagger \quad , \tag{222}$$

that are compatible since A is an hermitian matrix. The states given in Eq. (205) describe a vector boson and six adjoint scalars of the gauge group $U(N)$.

Let us now consider the Ramond sector of the open string attached to the D3 branes. The world-sheet parity Ω' acts on the massless states as follows ($s_i = \pm\frac{1}{2}$):

$$\Omega'|s_1,\, s_2,\, s_3,\, s_4;\, ij\rangle = -\left(\gamma_{\Omega'_3}\right)_{im}\left(\gamma_{\Omega'_3}^{-1}\right)_{nj}\left(\Gamma^4\ldots\Gamma^9\right)|s_1,\, s_2,\, s_3,\, s_4;\, nm\rangle \;, \tag{223}$$

where we have used the second equation in (148) and the fact that the action of Ω' is the same as that of $\tilde{\Omega}$. Remember that $s_0 = 1/2$ and the GSO projection imposes $\sum_{i=1}^4 s_i =$ odd. Using the previous equations we get:

$$\left(\Gamma^4\ldots\Gamma^9\right)|s_1,\, s_2,\, s_3,\, s_4;\, nm\rangle = -i(2s_2)\cdot(2s_3)\cdot(2s_4)|s_1,\, s_2,\, s_3,\, s_4;\, nm\rangle =$$

$$= 2is_1|s_1,\, s_2,\, s_3,\, s_4;\, nm\rangle \quad , \tag{224}$$

where the GSO condition has been imposed. Then Eq. (223) becomes:

$$\Omega'|s_1,\, s_2,\, s_3,\, s_4;\, ij\rangle = -2is_1 \left(\gamma_{\Omega'_3}\right)_{im} \left(\gamma^{-1}_{\Omega'_3}\right)_{nj} |s_1,\, s_2,\, s_3,\, s_4;\, nm\rangle \quad . \tag{225}$$

Notice that $(2s_1)$ coincides with the eigenvalue of four-dimensional chirality defined as:

$$\Gamma_5 = i\,\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = 4\,N_0N_1 = 2\,N_1 \quad . \tag{226}$$

Then Eq.. (225) becomes

$$\Omega'|s_1,\, s_2,\, s_3,\, s_4;\, ij\rangle = -i\Gamma_5 \left(\gamma_{\Omega'_3}\right)_{im} \left(\gamma^{-1}_{\Omega'_3}\right)_{nj} |s_1,\, s_2,\, s_3,\, s_4;\, nm\rangle \quad . \tag{227}$$

The invariance of the state under Ω' yields the condition:

$$\lambda_\psi = -i\,\Gamma_5 \gamma_{\Omega'_3} \lambda^T_\psi \gamma^{-1}_{\Omega'_3} \quad , \tag{228}$$

where we have denoted by Γ_5 the four-dimensional chirality of the spinor.

Furthermore, writing

$$\lambda_\psi = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{229}$$

and imposing the hermiticity condition $\lambda_\psi = \lambda^\dagger_\psi$, we get $C = B^\dagger$.

Eq. (228), together with the explicit expression of the action of the world-sheet parity on the Chan-Paton factors given in Eq. (217), gives:

$$B = \epsilon\Gamma_5 B^T \qquad B^\dagger = -\epsilon\Gamma_5 \bar{B} \quad , \tag{230}$$

where $\bar{B}$ denotes the complex conjugate of B. The last two conditions are compatible if one remembers that the four-dimensional chirality changes its sign under the complex conjugate operation.

The matrix B is antisymmetric if $\Gamma_5 = -\epsilon$, while is symmetric if $\Gamma_5 = \epsilon$. These conditions, and hence the open string spectrum, are exactly the same as the ones given in Ref. [32]. Consistently with that, Eq. (220) implies B to transform under a gauge transformation as follows

$$B' = UBU^T \quad , \tag{231}$$

which is the appropriate transformation property for the two-index symmetric and antisymmetric representation. In conclusion the spectrum of the massless states of type $0'$ theory consists of one gluon, six real scalars, all in the adjoint representation of the gauge group $SU(N)$, two Dirac fermions in the two-index antisymmetric and two Dirac fermions in the two-index symmetric representation of the gauge group. We have two Dirac fermions for instance in the antisymmetric representation of the gauge group $SU(N)$ because the state in Eq. (225), after taking into account the GSO projection, has four degrees of freedom with four-dimensional chirality $+1$ and four degrees of freedom with four-dimensional

chirality -1. Notice that, even if the world-volume gauge theory is non-supersymmetric, its spectrum satisfies the Bose-Fermi degeneracy condition. Finally the coefficient of one-loop β-function is given by:

$$\beta(g) = \frac{g^3}{(4\pi)^2}\left[-\frac{11}{3}N + 6\frac{N}{6} + 2\frac{4}{3}\left(\frac{N-2}{2} + \frac{N+2}{2}\right)\right] = 0 \tag{232}$$

and therefore the gauge theory living on a D3 brane of type $0'$ theory is conformal invariant at least at one-loop.

6.5 One-loop vacuum amplitude

In type $0'$ string theory the free energy is simply:

$$Z^o = Z_1^o + Z_{\Omega'}^o = 2\int_0^\infty \frac{d\tau}{2\tau}\mathrm{Tr}_{\mathrm{NS,\,R}}\left[e^{-2\pi\tau L_0}(-1)^{G_{bc}}P_{\mathrm{GSO}}P_{(-1)^{F_s}}\left(\frac{1+\Omega'}{2}\right)\right], \tag{233}$$

where the factor 2 in front of the last equation takes into account the two orientations of the open string exchanged in the loop. The annulus contribution corresponds to the interaction between two Dp branes while the Möbius strip describes the interaction between the Dp brane and the O9-plane.

Let us compute it explicitly. Z_1^o is obtained by multiplying the expression in Eq. (137) with $N = M$ by a factor $1/2$ due to the orientifold projection, obtaining

$$\begin{aligned} Z_1^o &= 2\int_0^\infty \frac{d\tau}{2\tau}\mathrm{Tr}_{\mathrm{NS,\,R}}\left[e^{-2\pi\tau L_0}P_{\mathrm{GSO}}(-1)^{G_{bc}}P_{(-1)^{F_s}}\frac{1}{2}\right] \\ &= N^2 V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}}\int_0^\infty \frac{d\tau}{\tau^{\frac{p+3}{2}}}e^{-\frac{y^2\tau}{2\pi\alpha'}} \\ &\times \frac{1}{2}\left[\left(\frac{f_3(k)}{f_1(k)}\right)^8 - \left(\frac{f_4(k)}{f_1(k)}\right)^8 - \left(\frac{f_2(k)}{f_1(k)}\right)^8\right] \quad , \end{aligned} \tag{234}$$

that in the closed string channel ($\tau = 1/t$) becomes:

$$Z_1^c = N^2 V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}}\int_0^\infty \frac{dt}{t^{\frac{9-p}{2}}}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{1}{2}\left[\left(\frac{f_3(q)}{f_1(q)}\right)^8 - \left(\frac{f_2(q)}{f_1(q)}\right)^8 - \left(\frac{f_4(q)}{f_1(q)}\right)^8\right]. \tag{235}$$

In order to compute $Z_{\Omega'}^o$ let us observe that:

$$\mathrm{Tr}\left[e^{-2\pi\tau(N_X+N_{bc})}\Omega'\right] = (ik)^{2/3}2^{\frac{9-p}{2}}f_2^{p-9}(ik)f_1^{1-p}(ik)\cdot k^{-2} \tag{236}$$

$$\mathrm{Tr}_{\mathrm{NS}}\left[e^{-2\pi\tau(N_\psi+N_{\beta\gamma})}\Omega'\right] = -i(ik)^{1/3}f_3^{9-p}(ik)f_4^{p-1}(ik)\cdot k \tag{237}$$

$$\mathrm{Tr}_{\mathrm{NS}}\left[e^{-2\pi\tau(N_\psi+N_{\beta\gamma})}\Omega'(-1)^F\right] = i(ik)^{1/3}f_3^{p-1}(ik)f_4^{9-p}(ik)\cdot k \quad , \tag{238}$$

where N is the world-sheet number operator defined in each sector of the string Fock spaces and the contribution to the traces coming from the ghost and superghost is already included. Instead the trace over the Chan-Paton factors gives:

$$\begin{aligned} \mathrm{Tr}^{\mathrm{C.P.}}\left[\langle hk|\Omega'|ij\rangle\right] &= \delta_{ik}\delta_{hj}\left(\gamma_{\Omega'_p}\right)_{im}\langle hk|nm\rangle\left(\gamma^{-1}_{\Omega'_p}\right)_{nj} \\ &= \mathrm{Tr}\left[\gamma^T_{\Omega'_p}\gamma^{-1}_{\Omega'_p}\right] = -2\,N\,e^{i\pi\left(\frac{5-p}{4}\right)}\delta_{p,(1,5,9)} \quad , \end{aligned} \tag{239}$$

where we have used Eq.s (202) and (215) and the normalization $\langle hk|nm\rangle = \delta_{kn}\delta_{hm}$. Notice that this definition of the trace gives also the correct result for the planar diagram. In fact we get:

$$\mathrm{Tr}^{\mathrm{C.P.}}\left[\langle hk|ij\rangle\right] = \mathrm{Tr}\left[\lambda\lambda\right] = \delta_{ki}\delta_{hj}\langle hk|ij\rangle = \delta_{ii}\delta_{jj} = (2N)^2 \quad . \tag{240}$$

Furthermore, remembering the action of the space-time fermion number operator on the Chan-Paton factors, given in Eq.s (126) and (127), we can write:

$$\mathrm{Tr}^{\mathrm{C.P.}}\left\langle hk|(-1)^{F_s}\Omega'|ij\right\rangle = \mathrm{Tr}\left[\gamma^{-1}_{\Omega'_p}\gamma^{-1}_{(-1)^{F_s}}\gamma^{T}_{\Omega'_p}\gamma^{T}_{(-1)^{F_s}}\right] = -\mathrm{Tr}\left[\gamma^{-1}_{\Omega'_p}\gamma^{T}_{\Omega'_p}\right] \quad . \tag{241}$$

For the Möbius diagram, in the NS sector, we get the following expression:

$$\begin{aligned} Z^{o}_{NS;\,\Omega'} &= \int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}_{\mathrm{NS}}\left[e^{-2\pi\tau L_0}(-1)^{G_{bc}}P_{\mathrm{GSO}}P_{(-1)^{F_s}}\frac{\Omega'}{2}\right] \\ &= \left(-2\,Ne^{i\pi\left(\frac{5-p}{4}\right)}\delta_{p,\,(1,5,9)}\right)\frac{V_{p+1}}{8}(8\pi^2\alpha')^{-\frac{p+1}{2}}\left[\tilde{Z}^{op.}_{\Omega'}+\tilde{Z}^{op.}_{\Omega'(-1)^{F_s}}\right], \end{aligned} \tag{242}$$

with:

$$\begin{aligned} \tilde{Z}^{o}_{\Omega'} &= -\tilde{Z}^{o}_{\Omega'(-1)^{F_s}} = 2^{\frac{9-p}{2}}\int_0^\infty \frac{d\tau}{\tau^{(p+3)/2}}e^{-\frac{y^2\tau}{2\pi\alpha'}} \\ &\times \left[\left(\frac{f_3(ik)}{f_2(ik)}\right)^{9-p}\left(\frac{f_4(ik)}{f_1(ik)}\right)^{p-1} - \left(\frac{f_4(ik)}{f_2(ik)}\right)^{9-p}\left(\frac{f_3(ik)}{f_1(ik)}\right)^{p-1}\right] \;, \end{aligned} \tag{243}$$

where Eq.s (236), (237), (238), (239) and (241) have been taken into account. By using Eq. (243) in Eq. (242) we get:

$$Z^{o}_{NS;\,\Omega'} = 0 \quad . \tag{244}$$

It may be helpful to give Eq. (243) also in the closed string channel by performing the modular transformation $\tau = 1/(4\,t)$, obtaining:

$$\begin{aligned} &\tilde{Z}^{c}_{\Omega'} = 2^6 e^{i\pi\left(\frac{5-p}{4}\right)}\int_0^\infty dt e^{-\frac{y^2}{8\pi\alpha' t}} \\ &\times\left[\left(\frac{f_3(iq)}{f_1(iq)}\right)^{p-1}\left(\frac{f_4(iq)}{f_2(iq)}\right)^{9-p} - \left(\frac{f_4(iq)}{f_1(iq)}\right)^{p-1}\left(\frac{f_3(iq)}{f_2(iq)}\right)^{9-p}\right] \;, \end{aligned} \tag{245}$$

where $q = e^{-\pi t}$ and the identities in Eq.s (365) and (366) have been used.

Because of Eq. (244), the total bosonic contribution to the free energy simply reduces to the first two terms of Eq. (234), and in particular the contribution of the bosonic massless states turns out to be:

$$Z^{o}_{1}(\text{bosonic massless}) \;=\; 8N^2V_{p+1}(8\pi^2\alpha')^{-\frac{p+1}{2}}\int_0^\infty \frac{d\tau}{\tau^{\frac{p+3}{2}}}e^{-\frac{y^2\tau}{2\pi\alpha'}} \;, \tag{246}$$

where the factor $8N^2$ in front of the previous expression counts the number of bosonic degrees of freedom of the gauge theory living on the brane.

In the Ramond sector, the term corresponding to the Möbius diagram is given by:

$$Z^o_{R;\Omega'} = -\int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr}_R \left[e^{-2\pi\tau L_0} (-1)^{G_{bc}} P_{\mathrm{GSO}} P_{(-1)^{F_s}} \frac{\Omega'}{2} \right] \quad . \tag{247}$$

It is easy to compute the traces over the non-zero modes, getting:

$$\mathrm{Tr}_R^{nzm} \left[e^{-2\pi\tau(N_\psi + N_{\beta\gamma})} \Omega' (-1)^{G_{\beta\gamma}} \right] = \frac{(ik)^{-2/3}}{2^{(p-1)/2}} f_2(ik)^{p-1} f_1(ik)^{9-p} \cdot k^2 \tag{248}$$

$$\mathrm{Tr}_R^{nzm} \left[e^{-2\pi\tau(N_\psi + N_{\beta\gamma})} \Omega' (-1)^{F} \right] = \frac{(ik)^{-2/3}}{2^{(9-p)/2}} f_1(ik)^{p-1} f_2(ik)^{9-p} \cdot k^2 \quad , \tag{249}$$

where N are the world-sheet number operators and the ghost and superghost non zero-mode contributions to the traces have been already taken into account. The trace over the zero modes is instead given by:

$$\mathrm{Tr}_R^{z.m.} \left[\Omega' (-1)^{G^0_{\beta\gamma}} \right] = -2^4 \delta_{9,p} \tag{250}$$

$$\mathrm{Tr}_R^{z.m.} \left[\Omega' (-1)^{F_0} \right] = -2^4 \delta_{p,1} \quad . \tag{251}$$

By collecting Eq.s (236), (248), (249), (250) and (251) we get:

$$\begin{aligned} Z^o_{R;\Omega'} = & \frac{V_{p+1}}{4} (8\pi^2\alpha')^{-\frac{p+1}{2}} \left(-2\, N e^{i\pi\left(\frac{5-p}{4}\right)} \delta_{p,\,(1\,,5\,,9)} \right) \int_0^\infty \frac{d\tau}{\tau} \tau^{-\frac{p+1}{2}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \\ & \times \left[\delta_{9,p} \left(\frac{f_2(ik)}{f_1(ik)} \right)^8 + 2^4 \delta_{p,1} \right] \quad . \end{aligned} \tag{252}$$

The previous expression can be written also in the closed channel as:

$$\begin{aligned} Z^c_{R;\Omega'} = & 2^4\, V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \left(-2 N e^{i\pi\left(\frac{5-p}{4}\right)} \delta_{p,\,(1\,,5\,,9)} \right) \int_0^\infty dt\, e^{-\frac{y^2}{8\pi\alpha' t}} \\ & \times \left[\delta_{9,p} \left(\frac{f_2(iq)}{f_1(iq)} \right)^8 + \delta_{p,1} \right] \quad . \end{aligned} \tag{253}$$

For $p = 3$ the fermionic free energy contribution reduces to the third term of Eq. (234) that in the field theory limit gives:

$$Z^o_1(\text{fermionic massless}) \quad = \quad -8N^2 V_{p+1} (8\pi^2\alpha')^{-\frac{p+1}{2}} \int_0^\infty \frac{d\tau}{\tau^{\frac{p+3}{2}}} e^{-\frac{y^2\tau}{2\pi\alpha'}} \quad . \tag{254}$$

The number $8N^2$ gives exactly the number of fermionic degrees of freedom of the world-volume gauge theory which indeed consist of two Dirac spinors in the two-index symmetric representation of the gauge group and two Dirac spinors in the two-index antisymmetric one.

For $p = 9$ Eq. (253) is divergent in the infrared limit $t \to \infty$. This divergence signals, as in type I string theory, the presence of a R-R tadpole which must be cancelled by introducing a suitable background of D9 branes. In order to determine it we compute the Klein-Bottle amplitude for this model. It is given by:

$$Z_{\mathrm{KB}} = \int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr} \left[\frac{\Omega'}{2} e^{-2\pi\tau(L_0 + \tilde{L}_0)} P_{\mathrm{GSO}} (-1)^{G_{bc} + \tilde{G}_{bc} + G_{\beta\gamma} + \tilde{G}_{\beta\gamma}} \right] \quad . \tag{255}$$

where P_{GSO} has been defined in Eq.(107).

By computing the trace over the oscillators one gets ($k = e^{-\pi\tau}$)

$$
\begin{aligned}
\mathrm{Tr}\left[\Omega' e^{-2\pi\tau(N_X+\tilde{N}_X+N_{bc}+\tilde{N}_{bc})}\right] &= \frac{k^{\frac{4}{3}}}{[f_1(k^2)]^8}\cdot k^{-4} \\
\mathrm{Tr}_{\mathrm{NS-NS}}\left[\Omega' e^{-2\pi\tau(N_\psi+\tilde{N}_\psi+N_{\beta\gamma}+\tilde{N}_{\beta\gamma})}\right] &= -k^{\frac{2}{3}}\left[f_4(k^2)\right]^8\cdot k^2 \\
\mathrm{Tr}_{\mathrm{NS-NS}}\left[\Omega' e^{-2\pi\tau(N_\psi+\tilde{N}_\psi+N_{\beta\gamma}+\tilde{N}_{\beta\gamma})}(-1)^{F+\tilde{F}}\right] &= -k^{\frac{2}{3}}\left[f_4(k^2)\right]^8\cdot k^2 \\
\mathrm{Tr}_{\mathrm{R-R}}\left[\Omega' e^{-2\pi\tau(N_\psi+\tilde{N}_\psi+N_{\beta\gamma}+\tilde{N}_{\beta\gamma})}\right] &= k^{-\frac{4}{3}} f_1^8(k^2)\cdot k^4 \\
\mathrm{Tr}_{\mathrm{R-R}}\left[\Omega' e^{-2\pi\tau(N_\psi+\tilde{N}_\psi N_{\beta\gamma}+\tilde{N}_{\beta\gamma})}(-1)^{F+\tilde{F}}\right] &= k^{-\frac{4}{3}} f_1^8(k^2)\cdot k^4 \ , \qquad (256)
\end{aligned}
$$

where we have already added the contribution coming from ghosts and superghosts. Let us consider now the zero modes contributions. In the bosonic sector one has:

$$
V_d \int \frac{d^d p}{(2\pi)^d} e^{-\alpha'\pi\tau p^2} = \frac{2^{d/2}\, V_d}{(8\pi^2\tau\alpha')^{d/2}} \quad . \qquad (257)
$$

By using Eq. (156), one can easily check that the R-R zero modes give a vanishing contribution to the trace in the partition function:

$$
\begin{aligned}
\mathrm{Tr}_{z.m.\,R-R}[\Omega'] &= \sum_{A,B} {}_{-1/2}<\tilde{C}|_{-1/2}<D|\Omega'|A>_{-1/2}|\tilde{B}>_{-1/2}(C)_{BC}(C)_{AD} \\
&= \mathrm{Tr}(\Gamma^{11}) = 0 \ , \qquad (258)
\end{aligned}
$$

where the following identity has been used:

$$
<A|B> = \left(C^{-1}\right)^{AB} \quad . \qquad (259)
$$

Thus collecting the contributions of the zero and non-zero modes we get:

$$
Z_{\mathrm{KB}} = -2^4 \frac{V_{10}}{(8\pi^2\alpha')^5}\int_0^\infty \frac{d\tau}{\tau^6}\left[\frac{f_4(k^2)}{f_1(k^2)}\right]^8 \quad . \qquad (260)
$$

It is useful to write Eq. (260) in the closed string channel by performing the modular transformation $\tau = 1/4t$. According to Eq. (364) we have ($q = e^{-\pi t}$):

$$
\begin{aligned}
Z_{\mathrm{KB}} &= -2^{10}\frac{V_{10}}{(8\pi^2\alpha')^5}\int_0^\infty dt \left[\frac{f_2(q^2)}{f_1(q^2)}\right]^8 \\
&= -2^{10}\frac{V_{10}}{(8\pi^2\alpha')^5}\int_0^\infty dt\, \frac{1}{2}\left[\frac{f_2(q)}{f_1(q)}\right]^8 \quad , \qquad (261)
\end{aligned}
$$

where the last identity follows after the change of variable $t \to t/2$.

The sum of the contributions from the Klein bottle in Eq. (261), the Möbius diagram in Eq. (253) for $p = 9$ and the second term in Eq. (235) again for $p = 9$ corresponding to the R-R spin structure is equal to:

$$
\frac{V_{10}}{(8\pi^2\alpha')^5}\int_0^\infty dt \left[-2^{10}\cdot\frac{1}{2}\left[\frac{f_2(q)}{f_1(q)}\right]^8 + 2^6 N\cdot\frac{1}{2}\left[\frac{f_2(iq)}{f_1(iq)}\right]^8 - N^2\cdot\frac{1}{2}\left[\frac{f_2(q)}{f_1(q)}\right]^8\right]. \qquad (262)
$$

If we restrict ourselves to the contribution of the massless states that is obtained by taking the ratio $(f_2/f_1)^8 = 2^4$, we get a R-R tadpole given by:

$$- \frac{V_{10}}{(8\pi^2\alpha')^5} \left[N - 32\right]^2 8 \int_0^\infty dt \quad , \tag{263}$$

that vanishes if we choose $N = 32$. This means that type $0'$ is free from R-R tadpoles if we have a background of 32 D9 branes [30] as in type I theory. Notice that, as previously discussed, had we chosen $\gamma_{\Omega'_9}$ to be an antisymmetric matrix, we would have obtained Eq. (215) with the opposite sign. This would have given a minus sign in front of the middle term in Eq.(262) and the cancellation of the R-R tadpole would not have been possible. The NS-NS tadpole can instead be extracted from the contribution of the massless states to the first and last terms of Eq. (235). It is given by:

$$N^2 \frac{V_{10}}{(8\pi^2\alpha')^5} 8 \int_0^\infty dt \quad , \tag{264}$$

that cannot be cancelled.[30]

Let us now compute the interaction between two Dp branes in the closed string channel, by using the boundary state formalism. But, before we do that, let us give the generalization of Eq. (252) to the case in which we have an Op' instead of an O9-plane orientifold. In this case Eq. (247) is modified as follows:

$$Z^o_{R;\Omega' I_{9-p'}} = - \int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr}_{\mathrm{R}} \left[\mathrm{e}^{-2\pi\tau \mathrm{L}_0} (-1)^{\mathrm{G_{bc}}} \mathrm{P_{GSO}}\, \mathrm{P}_{(-1)^{\mathrm{F_s}}} \frac{\Omega' \, \mathrm{I}_{9-\mathrm{p}'}}{2} \right] \quad , \tag{265}$$

where we have assumed that $p' \geq p$ and $I_{9-p'}$ is the inversion on $9 - p'$ coordinates, i.e:

$$I_{9-p'} \; : \; (x^{p'+1}, \ldots , x^9) \to (-x^{p'+1}, \ldots , -x^9) \quad . \tag{266}$$

The presence in the trace of the reflection operator follows from being the orientifold p'-plane obtained by performing $9-p'$ T-dualities on $O9$. In general, n T dualities transform Ω' into $\Omega' I_n (-1)^{F_L}$ being F_L the space time fermion number. The reason why the orientifold projector contains also the term $(-1)^{F_L}$ is because the operator $\Omega' I_{2n}$ squares to unity only for n even. In fact I_{2n}^2 represents a 2π rotation in n planes and, for n odd, is equal to $(-1)^{F_s} = (-1)^{F_L + F_R}$. Therefore:

$$\left[\Omega' I_{2n} (-1)^{F_L}\right]^2 = I_{2n}^2 (-1)^{F_L+F_R} = \mathbb{I} \qquad n = 1, 3 \quad , \tag{267}$$

where we have used the fact that $\Omega' (-1)^{F_L} \Omega'^{-1} = (-1)^{F_R}$ and $\Omega'^2 = 1$.

$(-1)^{F_L}$ gives an extra minus sign in the R-sector depending on being the open string considered respectively the right or left sector of closed string. However, this sign ambiguity is completely irrelevant in the discussion below and therefore we ignore it assuming for simplicity in the following that $(-1)^{F_L}$ does not act on the open string.

$I_{9-p'}$ acts on the NS vacuum as:

$$I_{9-p'} |0\rangle_{-1} = |0\rangle_{-1} \quad . \tag{268}$$

In the R sector, instead, in order to determine the action of the reflection operator on the vacuum, we observe that a reflection in a plane corresponds to a rotation of an angle $\pm\pi$, therefore we can write for p' odd:

$$I_{9-p'}|s_0\ldots s_4\rangle_{-1/2} = e^{\pm i\pi\sum_{i=(p'+1)/2}^{4}S^{2i,2i+1}}|s_0\ldots s_4\rangle_{-1/2} \quad , \tag{269}$$

being $S^{i,j}$ the zero modes of the Lorentz group generators, i.e:

$$S^{i,j} = -\frac{i}{2}[\psi_0^i,\, \psi_0^j] \quad , \quad \sqrt{2}\psi_0^i \equiv \Gamma^i \quad . \tag{270}$$

By introducing the operators N_i given in Eq. (63) it is straightforward to verify that $N_i \equiv S^{2i,2i+1}$. In conclusion we get:

$$\begin{aligned} I_{9-p'}|s_0\ldots s_4\rangle_{-1/2} &= \prod_{i=(p'+1)/2}^{4}(\pm 2iN_i)|s_0\ldots s_4\rangle_{-1/2} \\ &= (\pm)^{\frac{9-p'}{2}}\Gamma^{p'+1}\ldots\Gamma^9|s_0\ldots s_4\rangle_{-1/2} \quad , \end{aligned} \tag{271}$$

where we have taken into account that the state $|s_0\ldots s_4\rangle$ is an eigenstate of the operator N_i with eigenvalue $s_i = \pm 1/2$. We fix the sign ambiguity in the previous equation by observing that, as previously asserted, 9-p' T-dualities have to be equivalent to the action of $\Omega' I_{9-p'}$. This means that :

$$(T_{9-p'})^{-1}\Omega_9' T_{9-p'} = \Omega_{p'}' I_{9-p'} \Longrightarrow - = -(\pm)^{(9-p')/2}(-)^{(9-p')/2} \quad , \tag{272}$$

where, as explained before, we have neglected the action of $(-1)^{F_L}$ and denoted by Ω_A' the expression of the world sheet operator given in Eq. (148) taken with $A = 9$ and $A = p'$. From Eq. (272), we see that the action of $I_{9-p'}$ given in Eq. (271) is compatible with the action of Ω' given in Eq. (148), only if Eq. (271) is taken with the minus sign.

Eq.s (236), (248), (249), (250) and (251) are modified as follows (p and p' are odd, with $p' \geq p$):

$$\begin{aligned} \mathrm{Tr}\left[e^{-2\pi\tau(N_X+N_{bc})}\Omega' I_{9-p'}\right] &= (ik)^{2/3}2^{\frac{p'-p}{2}}f_2^{p-p'}(ik)f_1^{(p'-p)-8}(ik)\cdot k^{-2} \\ \mathrm{Tr}_R^{nzm}\left[e^{-2\pi\tau(N_\psi+N_{\beta\gamma})}\Omega' I_{9-p'}(-1)^{G_{\beta\gamma}}\right] &= \frac{(ik)^{-2/3}}{2^{[8-(p'-p)]/2}}f_2(ik)^{8-(p'-p)}f_1(ik)^{p'-p}\cdot k^2 \\ \mathrm{Tr}_R^{nzm}\left[e^{-2\pi\tau(N_\psi+N_{\beta\gamma})}\Omega' I_{9-p'}(-1)^{F}\right] &= \frac{(ik)^{-2/3}}{2^{(p'-p)/2}}f_1(ik)^{8-(p'-p)}f_2(ik)^{p'-p}\cdot k^2 \\ \mathrm{Tr}_R^{z.m.}\left[\Omega' I_{9-p'}(-1)^{G_{\beta\gamma}^0}\right] &= -2^4\delta_{p,p'} \\ \mathrm{Tr}_R^{z.m.}\left[\Omega' I_{9-p'}(-1)^{F_0}\right] &= -2^4\delta_{|p-p'|,8} \quad . \end{aligned} \tag{273}$$

The last two equations have been derived in Eq.s (407) and (408). Inserting Eq. (273) in Eq. (265) one gets:

$$\begin{aligned} Z^o_{R;\Omega' I_{9-p'}} &= \frac{V_{p+1}}{4(8\pi^2\alpha')^{\frac{p+1}{2}}}\mathrm{Tr}\left[\gamma^T_{\Omega' I_{9-p'}}\gamma^{-1}_{\Omega' I_{9-p'}}\right] \\ &\times \int_0^\infty \frac{d\tau}{\tau}\tau^{-\frac{p+1}{2}}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\delta_{p,p'}\left(\frac{f_2(ik)}{f_1(ik)}\right)^8 + 2^4\delta_{p',p+8}\right] \quad . \end{aligned} \tag{274}$$

where $\gamma_{\Omega' I_{9-p'}}$ is the matrix representing the orientifold action on the Chan-Paton factors. The explicit form of such a matrix can be determined by first observing that, in order to take into account the brane images under $\Omega' I_{9-p'}$, the Chan-Paton factors have to be $2N \times 2N$ matrices. This implies that also $\gamma_{\Omega' I_{9-p'}}$ has to be a $2N \times 2N$ matrix. Furthermore, following Ref. [80], $\gamma_{\Omega' I_{9-p'}}$ has to be symmetric or antisymmetric, i.e:

$$\mathrm{Tr}\left[\gamma^T_{\Omega' I_{9-p'}} \gamma^{-1}_{\Omega' I_{9-p'}}\right] = \pm 2N \quad . \tag{275}$$

We can rewrite Eq. (274) in the closed string channel by using Eq.s (365) getting:

$$\begin{aligned} Z^c_{R;\Omega' I_{9-p'}} &= 2^{p'-4} \frac{V_{p+1}}{2(8\pi^2\alpha')^{\frac{p+1}{2}}} \mathrm{Tr}\left[\gamma^T_{\Omega' I_{9-p'}} \gamma^{-1}_{\Omega' I_{9-p'}}\right] \\ &\times \int_0^\infty \frac{dt}{t^{(9-p')/2}}\, e^{-\frac{y^2}{8\pi\alpha' t}} \left[\delta_{p,p'} \left(\frac{f_2(iq)}{f_1(iq)}\right)^8 + \delta_{p',p+8}\right] \quad . \end{aligned} \tag{276}$$

The boundary state in type $0'$ string, as in type I theory, may be written as the sum of the usual boundary state describing a Dp brane and of the crosscap state which gives a microscopic description of the orientifold fixed plane. The Möbius contribution in the open string vacuum amplitude corresponds, in fact, in the closed channel to the interaction of the boundary with the crosscap. But, we have already seen that the contribution of the Möbius strip in the NS-NS sector is zero. Therefore we can write:

$$|Dp\rangle_{NS-NS} = |Bp\rangle_{NS-NS} \qquad |Dp\rangle_{R-R} = |Bp\rangle_{R-R} + |Cp'\rangle_{R-R} \ , \tag{277}$$

where the expression of the crosscap describing the orientifold p'-plane is for p' odd:

$$|Cp'\rangle_{R-R} = \frac{1}{2}\left[|Cp',+\rangle_{\mathrm{R-R}} + (-1)^{\frac{9-p'}{2}} |Cp',-\rangle_{\mathrm{R-R}}\right], \tag{278}$$

with [21]

$$|Cp',\eta\rangle = |Cp'_X\rangle\, |Cp'_\psi,\eta\rangle_{\mathrm{R-R}}\, |Cp'_{gh,\,sgh}\rangle \tag{279}$$

$$|Cp_X\rangle = 2^{p'-4}\frac{\hat{T}_{p'}}{2} \prod_{n=1}^\infty e^{-\frac{(-1)^n}{n}\alpha_{-n}\cdot S\cdot\tilde{\alpha}_{-n}} \delta^{9-p'}(\hat{q}-y)|0\,,0\rangle \tag{280}$$

$$|Cp'_\psi,\eta\rangle_{R-R} = \prod_{n=1}^\infty e^{i\eta(-1)^n \psi_{-n}\cdot S\cdot\tilde{\psi}_{-n}} |Cp'_\psi,\eta\rangle^{(0)}_{R-R} \quad , \tag{281}$$

where $S_{\mu\nu} = (\eta_{\alpha\beta}, -\delta_{ij})$ with $\alpha,\beta = 0\ldots p'$, $i,j = p'+1\ldots 9$ and

$$|Cp'_\psi, 0\rangle^{(0)}_{\mathrm{R-R}} = \left[C\Gamma^0\ldots\Gamma^p\left(1 + i\eta\Gamma^{11}\right)\right]_{AB} |A\rangle|\tilde{B}\rangle \quad , \tag{282}$$

where the part of the boundary states depending on the ghost and superghost is the standard one with the modifications discussed in Sect. 6.3.

The normalization of $|Cp'_X\rangle$ can be fixed through the open/closed string duality, by requiring the amplitude ${}_{R-R}\langle Bp|\Delta|Cp'\rangle_{R-R} + {}_{R-R}\langle Cp'|\Delta|Bp\rangle_{R-R}$, where Δ is the closed string propagator, to reproduce Eq.(276). In particular, the normalization in Eq. (280) has

been obtained by taking in Eq. (276) the symmetric choice for the matrices representing the orientifold action on the Chan-Paton factors. There is, however, an argument that gives the normalization of the crosscap almost without making any calculation. In fact, by comparing for instance the second term of Eq. (234), corresponding to the NS $(-1)^F$ spin structure, with the second term of Eq. (235), corresponding to the RR spin structure, we can get the normalization factor $\frac{\hat{T}_{p'}}{2}$ where $\hat{T}_{p'} = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p'}$. Then the additional normalization factor present in boundary state for the crosscap in Eq. (280) can be obtained by comparing the first terms in Eq.s (274) and (276). In fact, by comparing these two terms with those considered above in Eq.s (234) and (235), we see that we need in the normalization of the crosscap an extra factor $[-2^{p'-4}]$ [14]. This factor precisely agrees with the one derived in Ref. [88] by observing that the R-R charge of the O9-plane is equal to -2^5 times the R-R charge of the D9 brane and that the R-R charge of the Op plane can be obtained from that of the O9-plane by performing $(9-p)$ T-dualities. After $(9-p)$ T-dualities the R-R charge of one of the 2^{9-p} orientifold Op-planes will be equal to $-2^5/2^{9-p} = -2^{p-4}$ of the R-R charge of a Dp brane that is precisely the factor that we have obtained by open/closed string duality [15]. At this point it is clear why we needed to choose the minus sign in Eq. (271). In fact, had we chosen the opposite sign, we would not have been able to cancel the phase appearing in the fourth equation in (273) and this would have obliged us to have this phase in the normalization of the crosscap in Eq. (280). But this is not acceptable because T-duality cannot change the sign of R-R charge.

From the previous equations we deduce that the boundary state describing the crosscap in the R-R sector can be formally obtained from the boundary of a Dp' brane where each oscillator (α_n, ψ_n etc.) is multiplied by a factor i^n and the overall factor $[-2^{p'-4}]$ must be added by hand. Keeping in mind these substitutions, we can write the interaction between the boundary state and the crosscap by slightly modifying the expressions that give the interaction between a Dp and a Dp' brane. We get:[7, 21]

$$ {}_{R-R}\langle Bp,\, \eta|\Delta|Cp',\, \eta'\rangle_{R-R} = 2^{p'-4}\; V_{\hat{p}+1}\frac{\hat{T}_p}{2}\frac{\hat{T}_{p'}}{2}\frac{\pi\alpha'}{2}\int_0^\infty dt\, e^{-\frac{y^2}{8\pi\alpha' t}}\,(2\pi^2 t\alpha')^{-(9-\tilde{p})/2}A^{(\eta,\,\eta')}, $$

with

$$ \begin{aligned} A^{(\eta,\,\eta')} \;&=\; 16\,\delta_{\eta\eta',1}\delta_{p,p'}\prod_{n=1}^{\infty}\left[\frac{\det\left(1+\eta\eta' S_1 S_2^T (ie^{-\pi t})^{2n}\right)}{\det\left(1-S_1S_2^T(ie^{-\pi t})^{2n}\right)}\frac{\left(1-(ie^{-\pi t})^{2n}\right)^2}{\left(1+\eta\eta'(ie^{-\pi t})^{2n}\right)^2}\right] \\ &\qquad +16\delta_{|p'-p|,8}\delta_{\eta\eta',-1} \;\;, \end{aligned} \tag{283} $$

where $\hat{p} \equiv min(p,p')$ and $\tilde{p} \equiv max(p,p')$.

We have now all the ingredients to compute the interaction between a Dp brane and a crosscap. It is given by:

$$ \begin{aligned} Z^c \;&\equiv\; {}_{\mathrm{R-R}}\langle Bp|\Delta|Cp'\rangle_{\mathrm{R-R}} + {}_{\mathrm{R-R}}\langle Cp'|\Delta|Bp\rangle_{\mathrm{R-R}} \\ &=\; {}_{\mathrm{R-R}}\langle Bp|\Delta|Cp'\rangle_{\mathrm{R-R}} + {}_{\mathrm{R-R}}\overline{\langle Bp|\Delta|Cp'\rangle}_{\mathrm{R-R}} \;\;. \end{aligned} \tag{284} $$

[14] The minus sign can be obtained by comparing Eq. (281) with the corresponding one for a D brane given in Eq. (7.232) of Ref. [57].

[15] We thank J.F. Morales for explaining their derivation [88] to us.

We have also taken into account that, due to the structure of the boundary state, the only non-zero contribution comes from the R-R sector.

By using Eq. (283) we can separately compute the two contributions with $p = p'$ and $|p - p'| = 8$

$$
\begin{aligned}
{}_{\rm R-R}\langle Bp,\, \eta|\Delta|Cp',\, \eta'\rangle_{\rm R-R} &= 2^{(p-4)} \frac{V_{p+1}\, N}{(8\pi^2\alpha')^{\frac{p+1}{2}}} \,\delta_{p,p'}\delta_{\eta\eta',1} \\
&\times \int_0^\infty \frac{dt}{t^{\frac{9-p}{2}}}\, e^{-\frac{y^2}{8\pi\alpha' t}} \left(\frac{f_2(iq)}{f_1(iq)}\right)^8
\end{aligned}
\tag{285}
$$

and

$$
\begin{aligned}
{}_{\rm R-R}\langle Bp,\, \eta|\Delta|Cp',\, \eta'\rangle_{\rm R-R} &= -2^{\frac{p+3p'}{2}-\tilde{p}} \frac{V_{\hat{p}+1}\, N}{(8\pi^2\alpha')^{\frac{p+p'-\tilde{p}+1}{2}}} \,\delta_{|p-p'|,8}\delta_{\eta\eta',-1} \\
&\times \int_0^\infty \frac{dt}{t^{\frac{9-\tilde{p}}{2}}}\, e^{-\frac{y^2}{8\pi\alpha' t}} \quad .
\end{aligned}
\tag{286}
$$

In the last two equations we have used the identity:

$$
\frac{\hat{T}_{p'}}{2}\,\frac{\hat{T}_p}{2}\,\frac{\pi\alpha'}{2}(2\pi^2\alpha')^{-(9-\tilde{p})/2} = 2^{\frac{p+p'}{2}-\tilde{p}}(8\pi^2\alpha')^{-(p'+p-\tilde{p}+1)/2} \quad .
\tag{287}
$$

Furthermore, as a consequence of the identities:

$$
\begin{aligned}
&\overline{f}_1(iq) = e^{-i\frac{\pi}{12}} f_1(iq) \qquad \overline{f}_2(iq) = e^{-i\frac{\pi}{12}} f_2(iq) \\
&\overline{f}_3(iq) = e^{i\frac{\pi}{24}} f_4(iq) \qquad \overline{f}_4(iq) = e^{i\frac{\pi}{24}} f_3(iq) \quad ,
\end{aligned}
\tag{288}
$$

we have:

$$
{}_{\rm R-R}\langle Bp,\, \eta|\Delta|Cp',\, \eta'\rangle_{\rm R-R} = {}_{\rm R-R}\langle Cp',\, \eta|\Delta|Bp,\, \eta'\rangle_{\rm R-R} \;,
\tag{289}
$$

which allows us to write:

$$
\begin{aligned}
Z^c &= \frac{1}{2}\Big[{}_{\rm R-R}\langle Bp,\, +|\Delta|Cp',\, +\rangle_{\rm R-R} + (-1)^{9-(p+p')/2}{}_{\rm R-R}\langle Bp,\, -|\Delta|Cp',\, -\rangle_{\rm R-R}\Big] \\
&+\frac{1}{2}\Big[(-1)^{(9-p')/2}\,{}_{\rm R-R}\langle Bp,\, +|\Delta|Cp',\, -\rangle_{\rm R-R} \\
&+ (-1)^{(9-p)/2}\,{}_{\rm R-R}\langle Bp,\, -|\Delta|Cp',\, +\rangle_{\rm R-R}\Big] \\
&= 2^{\frac{p+3p'}{2}-4-\tilde{p}} \times \frac{V_{\hat{p}+1}\, N}{(8\pi^2\alpha')^{\frac{p+p'-\tilde{p}+1}{2}}} \\
&\times \int_0^\infty \frac{dt}{t^{\frac{9-\tilde{p}}{2}}}\, e^{-\frac{y^2}{8\pi\alpha' t}} \left[\delta_{p,p'}\left(\frac{f_2(iq)}{f_1(iq)}\right)^8 + 2^4\delta_{|p-p'|,8}\right] \quad ,
\end{aligned}
\tag{290}
$$

which reproduces Eq. (276), taken with $\gamma_{\Omega' I_{9-p'}}$ symmetric, for $p' \geq p$.

In the following we compute the interaction between two crosscaps. It can be easily extracted from Eq. (283) and one gets:

$$
\begin{aligned}
{}_{R-R}\langle Cp',\, \eta|\Delta|Cp',\, \eta'\rangle_{R-R} &= -2^{2(p'-4)}V_{p'+1}\left(\frac{\hat{T}_{p'}}{2}\right)^2 \frac{\pi\alpha'}{2}\delta_{\eta\eta',1} \\
&\times \int_0^\infty \frac{16\delta_{\eta\eta',1}dt}{(2\pi^2 t\alpha')^{(9-p')/2}} \prod_{n=1}^\infty \left(\frac{1+q^{2n}}{1-q^{2n}}\right)^8 \quad ,
\end{aligned}
\tag{291}
$$

that after the transformation $t \rightarrow 2t$ and the use of the identity given in Eq. (287) with $p = p'$, it can be rewritten as follows:

$$
{}_{R-R}\langle Cp',\, \eta|\Delta|Cp',\, \eta'\rangle_{R-R} = -2^{5(p'-5)/2}\, \frac{2V_{p'+1}\delta_{\eta\eta',1}}{(8\pi^2\alpha')^{(p'+1)/2}} \times \int_0^\infty \frac{dt}{t^{(9-p')/2}} \left[\left(\frac{f_2(q^2)}{f_1(q^2)} \right)^8 \right] \quad . \tag{292}
$$

Taking into account Eq. (278) we finally get:

$$
{}_{R-R}\langle Cp'|\Delta|Cp'\rangle_{R-R} = -2^{5(p'-5)/2} V_{p'+1} (8\pi^2\alpha')^{-(p'+1)/2} \times \int_0^\infty \frac{dt}{t^{(9-p')/2}} \left[\left(\frac{f_2(q^2)}{f_1(q^2)} \right)^8 \right] \quad , \tag{293}
$$

that transformed in the open string channel ($t = 1/(4\tau)$) becomes:

$$
Z_{KB} = -2^{(3p'-19)/2} \frac{V_{p'+1}}{(8\pi^2\alpha')^{(p'+1)/2}} \int_0^\infty \frac{d\tau}{\tau^{(p'+3)/2}} \left[\left(\frac{f_4(k^2)}{f_1(k^2)} \right)^8 \right] \quad . \tag{294}
$$

For $p' = 9$ it reproduces Eq. (260). The previous equation can also be derived in the open string channel by computing the quantity:

$$
Z_{KB} = \int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr} \left[\frac{\Omega' I_{9-p'}}{2} e^{-2\pi\tau(L_0+\tilde{L}_0)} (-1)^{G_{bc}+\tilde{G}_{bc}+G_{\beta\gamma}+\tilde{G}_{\beta\gamma}} P_{GSO} \right] \quad , \tag{295}
$$

where the GSO projection is defined in Eq. (107). One must use again Eq.s (256), while Eq. (257) becomes

$$
\int \frac{d^{10}p}{(2\pi)^{10}}\, Tr[\Omega' I_{9-p'} e^{-\alpha'\pi\tau\hat{p}^2}] = \int \frac{d^{p'+1}p}{(2\pi)^{p'+1}} \delta^{p'+1}(0) e^{-\alpha'\pi\tau p^2} \times \int d^{9-p'}p_\perp \delta^{9-p'}(2p_\perp) \quad , \tag{296}
$$

where the presence of two different delta functions is due to the action of the parity operator $I_{9-p'}$ that changes sign to $9 - p'$ components of the momentum. By computing the integrals we get:

$$
\int \frac{d^{10}p}{(2\pi)^{10}}\, Tr[\Omega' I_{9-p'} e^{-\alpha'\pi\tau\hat{p}^2}] = \frac{2^{(p'+1)/2}\; 2^{p'-9}\, V_{p'+1}}{(8\pi^2\tau\alpha')^{(p'+1)/2}}; \qquad V_{p'+1} \equiv (2\pi)^{p'+1}\delta^{p'+1}(0) \quad . \tag{297}
$$

For what concerns the R-R zero modes they are identically zero. Indeed, by using Eq. (156), together with the action of $I_{9-p'}$ on the zero modes given in Ref. [87] and that generalize Eq. (271) to the closed string case, one gets for p' odd:

$$
I_{9-p'} \left(|A\rangle_{-\frac{1}{2}} \otimes |\tilde{B}\rangle_{-\frac{1}{2}} \right) = \left(\Gamma^{11}\Gamma^9 ... \Gamma^{11}\Gamma^{p'+1} \right)^A{}_C \times \left(\Gamma^{11}\Gamma^9 ... \Gamma^{11}\Gamma^{p'+1} \right)^B{}_D |C\rangle_{-\frac{1}{2}} \otimes |\tilde{D}\rangle_{-\frac{1}{2}} \quad , \tag{298}
$$

and

$$\Omega' I_{9-p'}\left(|A\rangle_{-\frac{1}{2}}\otimes|\tilde{B}\rangle_{-\frac{1}{2}}\right)=\left(\Gamma^9...\Gamma^{p'+1}\right)^A_{\ C}\left(\Gamma^0...\Gamma^{p'}\right)^B_{\ E}|E\rangle_{-\frac{1}{2}}\otimes|\tilde{C}\rangle_{-\frac{1}{2}}\quad . \tag{299}$$

Thus evaluating the trace in the partition function, as showed in Appendix B, yields:

$$\mathrm{Tr}_{z.m.\,R-R}[\Omega' I_{9-p'}]=\mathrm{Tr}_{z.m.\,R-R}[\Omega' I_{9-p'}(-1)^{F+\tilde{F}}]=0\quad . \tag{300}$$

Thus collecting the contributions of the zero and non-zero modes we get Eq. (294).

We conclude this subsection by noticing that the one-loop open string diagrams involving a D3 brane dressed with an external gauge field and a stack of N undressed D3 branes get a non-vanishing contribution only from the annulus diagram (the Möbius strip is vanishing for $p=3$ as it follows from Eq.s (244) and (252)) and it is exactly equal to the one given in Eq. (40).

6.6 Type $0'$ theory in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$.

We have seen in Eq. (232) that the one-loop β-function of the world-volume gauge theory supported by a stack of N D3 branes in type $0'$ string theory is zero. For this reason, such a brane configuration does not represent an interesting model for exploring the validity of the gauge/gravity correspondence. In the following, in order to consider non-supersymmetric gauge theories having a non vanishing one-loop running coupling constant, we study type $0'$ string theory on the orbifold $\mathbb{R}^{1,5}\times\mathbb{C}^2/\mathbb{Z}_2$ and in this background we consider fractional D3 branes sitting at the orbifold singularity. The $\mathbb{Z}_2$ group has been already introduced in Sect. 3.2. The gauge theory living on the fractional D3 branes is the $\mathbb{Z}_2$ invariant subsector of the open string spectrum introduced in Sect. 6.4. In particular, by using the orbifold actions defined in Eq.s (17) and (142), it is possible to see that this spectrum, even if non-supersymmetric, satisfies the Bose-Fermi degeneracy condition. Indeed it contains an $SU(N)$ gauge field, two scalars in the adjoint representation of the gauge group, one Dirac fermion in the two-index symmetric and one Dirac fermion in the two-index antisymmetric representation of the gauge group. The one-loop β-function is therefore given by:

$$\beta(g)=\frac{g^3}{(4\pi)^2}\left[-\frac{11}{3}N+2\frac{N}{6}+\frac{4}{3}\left(\frac{N-2}{2}+\frac{N+2}{2}\right)\right]=-2N\frac{g^3}{(4\pi)^2}\quad , \tag{301}$$

that coincides with the β-function of $\mathcal{N}=2$ Super-Yang Mills.

In order to check the gauge/gravity correspondence in this case, one can apply the usual strategy, evaluating the one-loop vacuum amplitude of an open string stretching between a stack of N D3 fractional branes and a further D3 brane dressed with an $SU(N)$ background gauge field. Notice however that the one-loop vacuum amplitude, given by

$$\begin{aligned} Z\ &=\int_0^\infty\frac{d\tau}{\tau}\left[P_{(-1)^{F_s}}\left(\frac{1+\Omega'}{2}\right)\left(\frac{e+h}{2}\right)e^{-2\pi\tau L_0}(-1)^{G_{bc}}P_{\rm GSO}\right]\\ &\equiv\ Z^o_e+Z^o_{\Omega'}\quad , \end{aligned} \tag{302}$$

simply reduces to the annulus contribution Z^o_e . Indeed the Möbius strip $Z^o_{\Omega'}$ is zero because of the trace on the Chan-Paton factors. In fact, by using the definition of trace in Eq.

(239), taken in this case with $j,h = 1,\ldots,N,N+2,\ldots 2N+1$ and $i,k = N+1, 2N+2$ [16] and specifying Eq.s (216) and (214) for $p=3$

$$\gamma_{\Omega'_3} = \begin{pmatrix} 0 & \beta \\ i\beta & 0 \end{pmatrix} \quad ; \quad \gamma^T_{\Omega'_3}\gamma^{-1}_{\Omega'_3} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \tag{303}$$

where each element of the previous matrix is actually an $(N+1)\times(N+1)$ matrix, we get

$$\mathrm{Tr}\left[\gamma^T_{\Omega'_3}\gamma^{-1}_{\Omega'_3}\right] = \sum_{i=N+1,\,2N+2}\left[\sum_{h=1}^{N} + \sum_{h=N+2}^{2N+1}\right]\left(\gamma^T_{\Omega'_3}\right)_{ih}\left(\gamma^{-1}_{\Omega'_3}\right)_{hi} = i - i = 0. \tag{304}$$

In conclusion the one-loop vacuum amplitude reduces to the annulus contribution:

$$Z = Z^o_e + Z^o_h \quad , \tag{305}$$

with Z^o_e and Z^o_h exactly given respectively in Eq. (40) and in Eq. (41). In fact, the expressions in Eq. (305) that follow from Eq. (302) without the term with Ω' have an additional factor 1/4 with respect to the analogous ones in Eq. (35). But, on the other hand, the Chan-Paton factor gives a factor $4N$ as we are now going to show. From Eq. (240) we get:

$$\begin{aligned} Tr[\langle ij|hk\rangle] &= \sum_{i,k=N+1,2N+2}\left[\sum_{j,h=1}^{N} + \sum_{j,h=N+2}^{2N+1}\right]\delta_{jh}\delta_{ik} \\ &= \mathrm{Tr}\left[\mathbb{I}_2\right]\times\mathrm{Tr}\left[\mathbb{I}_{2N}\right] = 4N \quad . \end{aligned} \tag{306}$$

Being the expression for the free energy Z in Eq. (302) exactly equal to the one of type IIB theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$, discussed in Sect. 3.2, it is clear that the β-function for the non-supersymmetric theory living on N D3 branes of type $0'$ theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ is equal to the one of $\mathcal{N}=2$ super Yang-Mills in agreement with Eq. (301). Moreover, as in the case of type IIB on $\mathbb{C}^2/\mathbb{Z}_2$, the only non trivial contribution to the gauge theory parameters comes from the massless states propagating in the annulus without threshold corrections. Analogously, in the closed channel the only non trivial contribution comes from massless closed string states propagating in the cylinder, without threshold corrections and it leads exactly to the right values for the gauge theory parameters at one loop.

7 Gauge/Gravity Correspondence in Type IIB on $\Omega' I_6(-1)^{F_L}$

In this section we study type 0B string theory in the orientifold $\Omega' I_6(-1)^{F_L}$, where Ω' is the world-sheet parity operator and F_L is the space-time fermion number operator in the left sector.

[16] Notice that we are interested in computing the one-loop vacuum amplitude of an open string stretching between the stack of the N undressed branes (labelled by $1,\ldots,N$, with their "Ω'-images" labelled by $N+2,\ldots,2N+1$) and the dressed brane (labelled by $N+1, 2N+2$).

This theory has a non trivial background made of an orientifold fixed plane which is, by definition, the set of the points left invariant by the combined action of Ω' and I_6. In our case such a plane is located at $x^4 = \cdots = x^9 = 0$.

The world-volume gauge theory of N D3 branes in this orientifold of type 0B is an example of $\mathcal{N} = 4$ orientifold field theory, [34] which is planar equivalent (i.e. equivalent in the limit $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed) to $\mathcal{N} = 4$ super Yang-Mills. After discussing the case of type 0B on $\Omega' I_6 (-1)^{F_L}$ on flat space, we consider some orbifolds of this orientifold and, within this framework, we analyze the world-volume gauge theory living on a stack of N fractional branes. We start with discussing the gauge theory living on N fractional D3 branes in the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. In the planar limit this theory, which is non-supersymmetric, shows some interesting common features with $\mathcal{N} = 2$ super Yang-Mills. Then we drive our attention to the more interesting case of the orbifold $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. Here the gauge theory living on N fractional branes is the one recently discussed by Armoni-Shifman-Veneziano, that for $N = 3$ reduces to QCD with one flavour. In Table 2 the spectrum of the world-volume gauge theory of N D3 branes in type $0\mathrm{B}/\Omega' I_6 (-1)^{F_L}$ and its orbifolds is summarized.

7.1 Open and closed string spectrum

Let us determine the spectrum of the massless open string states attached to N D3 branes at the orientifold plane. The massless open string states in type 0 are given by Eq.s (123), (124 and (125). On these states we have then to impose the orientifold projection and select only those states that are invariant under the action of $\Omega' I_6$. In the NS-sector we have (in the picture -1):

$$\Omega' I_6 \ \ \psi^{\alpha}_{-1/2}|0,k\rangle \to -\psi^{\alpha}_{-1/2}|0,k\rangle \quad , \quad \Omega' I_6 \ \ \psi^{i}_{-1/2}|0,k\rangle \to -\psi^{i}_{-1/2}|0,k\rangle \tag{307}$$

and therefore the invariant states satisfy the constraint:

$$\gamma_{\Omega' I_6} \lambda^T_{A,\phi} \gamma^{-1}_{\Omega' I_6} = -\lambda_{A,\phi} \quad . \tag{308}$$

By using Eq. (271) with $p' = 3$, we can write:

$$\begin{aligned} I_6 |s_0 \dots s_4\rangle = e^{\pm i\pi(s_2+s_3+s_4)} |s_0 \dots s_4\rangle &= \prod_{i=2}^{4} (\pm 2 i N_i) |s_0 \dots s_4\rangle \\ &= \pm \Gamma^4 \dots \Gamma^9 |s_0 \dots s_4\rangle \quad . \end{aligned} \tag{309}$$

As previously asserted, we have to take the minus sign in Eq. (309) and therefore in the R-sector we have:

$$\Omega' I_6 |s_0 \dots s_4\rangle = -|s_0 \dots s_4\rangle \Longrightarrow \gamma_{\Omega' I_6} \lambda^T_{\psi} \gamma^{-1}_{\Omega' I_6} = -\lambda_{\psi} \quad . \tag{310}$$

In the previous equation we should also consider the action of the operator $(-1)^{F_L}$. This is irrelevant or gives an extra minus sign in the R-sector depending on being the open string

considered respectively the right or left sector of the closed string. However, it is simple to check that this sign ambiguity is completely irrelevant in determining the spectrum of the massless states.

In the last part of this section we determine the orientifold action on the Chan-Paton factors. First we observe that these have to be $2N\times 2N$ matrices in order to take into account their images under $\Omega' I_6$. Furthermore, following Ref. [80], they have to satisfy the constraint $\gamma_{\Omega' I_6} = \pm\gamma^T_{\Omega' I_6}$ that implies

$$\gamma_{\Omega' I_6} = \begin{pmatrix} 0 & \mathbb{I}_{N\times N} \\ \pm\mathbb{I}_{N\times N} & 0 \end{pmatrix}. \tag{311}$$

By substituting Eq. (311) in Eq.s (308) and (310), one gets for the bosonic and fermionic Chan-Paton factors the following expressions:

$$\lambda_{A,\phi} = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \qquad \lambda_{\psi} = \begin{pmatrix} 0 & B \\ \pm B^* & 0 \end{pmatrix}, \tag{312}$$

where in the last one we have implemented the hermiticity of the Chan-Paton factors and the matrix B can be chosen to be either symmetric or antisymmetric depending on how the sign in Eq. (311) is chosen. In particular the symmetric choice for the matrices given in Eq. (311) leads to fermion in the antisymmetric representation of the gauge group, while the antisymmetric one to fermions in the symmetric representation. The number of bosonic degrees of freedom is $8N^2$ which corresponds to one gauge boson and six real scalars transforming according to the adjoint representation of $SU(N)$. In the fermionic sector one has $8N^2 \pm 8N$ corresponding to four Dirac fermions in the two-index symmetric $(+)$ or antisymmetric $(-)$ representation. Notice that the spectrum does not satisfy the Bose-Fermi degeneracy condition that holds in type 0 and $0'$ theories. In this case such a degeneracy is present only in the large N limit.

Moreover the spectrum of this theory has the same bosonic content as $\mathcal{N}=4$ SYM. This is an example of planar equivalence [31, 34] between a supersymmetric model, the $\mathcal{N}=4$ SYM, which plays the role of the *parent* theory and a non-supersymmetric one, that is the orientifold $\Omega' I_6(-1)^{F_L}$ of type 0B, which is the *daughter* theory, the two being equivalent in the large N limit. In Sect. 7.3, by using string techniques, we will explicitly see that in this limit the two theories have the same β-function.

Let us consider the closed string spectrum. Since Ω' leaves invariant the metric and the dilaton, while it changes sign to the Kalb-Ramond field, it is easy to see that in the NS-NS sector the orientifold projection selects the following states:

$$\phi,\ g_{\alpha\beta},\ g_{ij},\ B_{i\alpha} \quad \text{with} \quad \alpha,\beta = 0,\ldots,3 \quad i,j=4,\ldots,9 \ , \tag{313}$$

where ϕ, g and B are respectively the dilaton, graviton and Kalb-Ramond fields. In the R-R sector the states which are even under the orientifold projection are

$$(\text{R}+,\text{R}+) \quad \rightarrow \quad C_0,\ C_{\alpha i},\ C_{0123},\ C_{\alpha\beta ij},\ C_{ijhk}, \tag{314}$$

$$(\mathrm{R}-,\mathrm{R}-) \quad \rightarrow \quad \bar{C}_{\alpha\beta}, \;\; \bar{C}_{ij}, \;\; \bar{C}_{\alpha\beta\gamma i}, \;\; \bar{C}_{\alpha ijk} \quad . \tag{315}$$

The previous results follow because, as explained at the end of Sec. 2, Ω' leaves C_2, $\bar{C}_0$ and $\bar{C}_4$ invariant and changes the sign of $\bar{C}_2$, C_0 and C_4. Notice that the R-R 5-form field strength surviving the orientifold projection is the self-dual one ($dC_4 = {}^*dC_4$), while the anti-self dual one ($d\bar{C}_4 = -{}^*d\bar{C}_4$) is projected out.

7.2 One-loop vacuum amplitude

In this section we compute the one-loop vacuum amplitude of open strings stretching between two stacks of N D3 branes in the orientifold $\Omega' I_6$, the action of the operator $(-1)^{F_L}$ being irrelevant in the open string calculation, as previously discussed.

The one-loop open string amplitude gets two contributions, the annulus Z_e, which encodes the information about the interaction between the two stacks of branes, and the Möebius strip $Z_{\Omega' I_6}$ which instead describes the interaction of each stack of N D3 branes with the O3-plane:

$$\begin{aligned} Z^o &\equiv Z_e^o + Z_{\Omega' I_6}^o \\ &= \int_0^\infty \frac{d\tau}{\tau} Tr_{\mathrm{NS-R}} \left[\frac{e + \Omega' I_6}{2} \; P_{(-1)^{F_s}} \, (-1)^{G_{bc}} \, P_{GSO} e^{-2\pi\tau L_0} \right] \quad . \end{aligned} \tag{316}$$

The annulus contribution is equal to the one in Eq. (137) with $M = N$ and with an extra factor 1/2 due to the orientifold projection. For $p = 3$ we get:

$$Z_e^o = N^2 \frac{V_4}{(8\pi^2\alpha')^2} \int_0^\infty \frac{d\tau}{2\tau^3} \left[\left(\frac{f_3(k)}{f_1(k)} \right)^8 - \left(\frac{f_4(k)}{f_1(k)} \right)^8 - \left(\frac{f_2(k)}{f_1(k)} \right)^8 \right] \quad . \tag{317}$$

In particular it vanishes because of the abstruse identity and this signals the absence of any force among the branes. The contribution of the Möbius strip, corresponding to the insertion of $\Omega' I_6$ in the trace, is instead non-trivial. Let us first compute, for such a term, the trace over the Chan-Paton factors. By fixing the standard normalization $\langle hk|nm\rangle = \delta_{kn}\delta_{hm}$, one finds:

$$\begin{aligned} &\mathrm{Tr}^{\mathrm{C.P.}} \left[\langle hk|\Omega' I_6|ij\rangle \right] = \mathrm{Tr}\left[\gamma_{\Omega' I_6}^{-1} \gamma_{\Omega' I_6}^T \right] \\ &\mathrm{Tr}^{\mathrm{C.P.}} \left[\langle hk|\Omega' I_6 (-1)^{F_s}|ij\rangle \right] = \mathrm{Tr}\left[\gamma_{\Omega' I_6}^{-1} \gamma_{(-1)^{F_s}}^{-1} \gamma_{\Omega' I_6}^T \gamma_{(-1)^{F_s}}^T \right] \quad . \end{aligned} \tag{318}$$

Furthermore, from the explicit form of the matrices introduced in the last expression and given in Eq.s (127) and (311), it is straightforward to check that

$$\mathrm{Tr}\left[\gamma_{\Omega' I_6}^{-1} \gamma_{(-1)^{F_s}}^{-1} \gamma_{\Omega' I_6}^T \gamma_{(-1)^{F_s}}^T \right] = -\mathrm{Tr}\left[\gamma_{\Omega' I_6}^{-1} \gamma_{\Omega' I_6}^T \right] \quad . \tag{319}$$

This identity implies that the NS contribution to the free energy vanishes.

A non-vanishing contribution comes from the R sector, where the trace over the non-zero modes ($n.z.m.$) gives

$$\mathrm{Tr}_R^{n.z.m.}\left[e^{-2\pi\tau(N_\psi+N_{\beta\gamma})}\Omega' I_6(-1)^{G_{\beta\gamma}}\right] = \frac{(ik)^{-2/3}}{2^4} f_2^8(ik)\cdot k^2 \tag{320}$$

$$\mathrm{Tr}_R^{n.z.m.}\left[e^{-2\pi\tau(N_\psi+N_{\beta\gamma})}\Omega' I_6(-1)^{F}\right] = (ik)^{-2/3} f_1^8(ik)\cdot k^2, \tag{321}$$

while the trace over the zero modes ($z.m.$) is given by:

$$\mathrm{Tr}_R^{z.m.}\left[\Omega' I_6(-1)^{G^0_{\beta\gamma}}\right] = \mathrm{Tr}_R^{z.m.}\left[(-1)^{G_0}\right] = -2^4 \tag{322}$$

$$\mathrm{Tr}_R^{z.m.}\left[\Omega' I_6(-1)^{F_0}\right] = \mathrm{Tr}_R^{z.m.}\left[(-1)^{F_0}\right] = 0 \quad . \tag{323}$$

By inserting Eqs. (318), (319) and (320) $\div$ (323) in the term with $\Omega' I_6$ in Eq. (316), we get:

$$Z^o_{\Omega' I_6} = \frac{V_4}{4(8\pi^2\alpha')^2}\,\mathrm{Tr}\left[\gamma^T_{\Omega' I_6}\gamma^{-1}_{\Omega' I_6}\right]\int_0^\infty \frac{d\tau}{\tau^3}\left(\frac{f_2(ik)}{f_1(ik)}\right)^8 \quad , \tag{324}$$

where we should use that $\mathrm{Tr}\left[\gamma^T_{\Omega' I_6}\gamma^{-1}_{\Omega' I_6}\right] = \pm 2N$. Notice that, because of the Möbius strip contribution, the interaction between the N D3 branes and the O3-plane in this orientifold does not vanish.

Eq. (324) is finite in the infrared limit $\tau \to \infty$. In order to analyse its behaviour in the UV regime we perform the modular transformation $\tau = 1/4t$ which leads into the closed string channel. We get:

$$Z^c_{\Omega' I_6} = \frac{V_4}{4(8\pi^2\alpha')^2}\,\mathrm{Tr}\left[\gamma^T_{\Omega' I_6}\gamma^{-1}_{\Omega' I_6}\right]\int_0^\infty \frac{dt}{t^3}\left(\frac{f_2(iq)}{f_1(iq)}\right)^8 \quad . \tag{325}$$

The previous equation reproduces Eq. (290) with $p = p' = 3$ for the symmetric choice of the Chan-Paton factors. The solution obtained by taking antisymmetric Chan-Paton factors would correspond to define the crosscap with the opposite sign with respect to the one in Eq. (280).

It is interesting to observe that the latter expression is invariant under the open/closed string duality and therefore it is also finite in the limit $t \to \infty$. In conclusion Eq. (324) is well-defined both in the IR and UV regimes and such a property provides a first hint of the absence of R-R tadpoles in this orientifold. We will come back on this point later.

The Möbius amplitude, together with the third term of Eq. (317), gives the total fermionic contribution to the free-energy which at the massless level reduces to:

$$Z^o(\text{femionic massless}) = -(8N^2 \pm 8N)\frac{V_4}{(8\pi^2\alpha')^2}\int_0^\infty \frac{d\tau}{\tau^3} \quad . \tag{326}$$

As usual, the factor $(8N^2 \pm 8N)$ in front of the previous expression counts the number of the fermionic degrees of freedom of the world-volume gauge theory, which indeed agrees with the counting of the previous subsection. As already noticed, we do not have the same number of bosonic and fermionic degrees of freedom propagating in the loop and, in particular, the additional $\pm 8N$ fermionic term comes from the Möbius strip, which therefore

is responsible of spoiling the Bose-Fermi degeneracy of the theory.[33] This contribution is subleading in the large N limit.

Notice that Eq. (324), apart from the Chan-Paton factors and the substitution $k \to ik$, is 1/2 of the free energy, describing in the R-sector the interaction between two D3 branes in type IIB string theory. It is, by the way, also equal, through the previous substitutions, to the correspondent expression in type 0 theory, given in Eq. (133).

The existence of the orientifold plane can be a source of inconsistency of the background if it generates R-R tadpoles which are not properly cancelled out. In order to check the consistency of the background in the present case, we have to analyse the field theory behaviour of the two closed string diagrams involving the orientifold plane. These are the Möbius strip, that describes the interaction between the branes and the orientifold plane, and the Klein bottle, which gives the self-interaction of the orientifold plane. We have already evaluated the Möbius strip and seen that it does not lead to any tadpole. In order to be sure about the consistency of the background, we should also consider the Klein bottle. It can be obtained from Eq. (294) for $p' = 3$. One gets:

$$Z_{KB} = -\frac{2^{-5} V_4}{(8\pi^2\alpha')^2} \int_0^\infty \frac{d\tau}{\tau^3} \left(\frac{f_4(k^2)}{f_1(k^2)} \right)^8 . \tag{327}$$

In order to write the previous expression in the closed string channel, one has to perform the modular transformation $\tau \to \frac{1}{4t}$ and to use the modular transformation properties of the functions f_i obtaining:

$$Z_{KB} = -\frac{2^{-5} V_4}{(8\pi^2\alpha')^2} \int_0^\infty \frac{dt}{t^3} \left(\frac{f_2(q^2)}{f_1(q^2)} \right)^8 , \tag{328}$$

which is finite in the limit $t \to \infty$ leading to no R-R tadpoles! One can conclude that the O3-plane does not generate any R-R tadpole and therefore the background is perfectly consistent, as expected, being the space transverse to the orientifold plane non compact.

7.3 One-loop vacuum amplitude with an external field

Let us consider, in the open channel, the one-loop vacuum amplitude of an open string stretching between a D3 brane dressed with a constant $SU(N)$ gauge field and a stack of N undressed D3 branes. The gauge field is chosen as in Eq. (34). The presence of the external field modifies Eq. (324) as follows:

$$\begin{aligned} Z^o(F)_{\Omega' I_6} &= -\frac{2\,\mathrm{Tr}\left[\gamma^T_{\Omega' I_6} \gamma^{-1}_{\Omega' I_6}\right]}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{d\tau}{\tau} \mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}} \sin\pi\nu_f \sin\pi\nu_g \\ &\quad \times \frac{f_2^4(ik)\Theta_2\left(i\nu_f\tau|i\tau+1/2\right)\Theta_2\left(i\nu_g\tau|i\tau+1/2\right)}{f_1^4(ik)\Theta_1\left(i\nu_f\tau|i\tau+1/2\right)\Theta_1\left(i\nu_g\tau|i\tau+1/2\right)} . \end{aligned} \tag{329}$$

Notice that in the previous expression we should have put $y = 0$ because all branes are located at the orientifold point. However we keep $y \neq 0$ because it provides a natural infrared cutoff. Eq. (329) describes the Möbius strip with the boundary on the dressed

brane. The trace over the Chan-Paton factors gives ± 2 counting the dressed brane and its image under $\Omega' I_6$. The overall factor 2, instead, is a consequence of the fact that in the trace we have to sum over two different but equivalent open string configurations: the first one in which only the end-point of the string parametrized by the world-sheet coordinate $\sigma = 0$ is charged under the gauge group, and the other one in which the gauge charge is turned on instead at the other end-point at $\sigma = \pi$.

In the last part of this subsection, in order to explore the gauge/gravity correspondence in this non-supersymmetric model, we evaluate the threshold corrections to the running coupling constant.

The starting point is again Eq. (329) that we now expand up to the quadratic order in the gauge fields without performing any field theory limit (more details on the calculation are contained B obtaining ($k = e^{-\pi\tau}$)):

$$\frac{1}{g^2_{\rm YM}} = \pm \frac{1}{(4\pi)^2} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \left(\frac{f_2(ik)}{f_1(ik)}\right)^8 \left[\frac{1}{3\tau^2} + k\frac{\partial}{\partial k}\log f_2^4(ik)\right] \quad , \tag{330}$$

where the upper sign refers to the antisymmetric choice of the Chan-Paton factors while the lower sign to the symmetric one.

We can now perform the field theory limit corresponding to $\tau \to \infty$, $\alpha' \to 0$ keeping the quantity $\sigma = 2\pi\alpha'\tau$ fixed. In this way from Eq. (330) we get:

$$\frac{1}{g^2_{\rm YM}} = \mp \frac{16}{3(4\,\pi)^2} \int_{1/\Lambda^2}^\infty \frac{d\sigma}{\sigma} {\rm e}^{-\mu^2\,\sigma} \quad , \tag{331}$$

where Λ is an UV cut-off and $\mu = \frac{y}{2\pi\alpha'}$ is an IR one. By using Eq. (54) and adding the contribution of the tree diagrams we get the following expression for the running coupling constant:

$$\frac{1}{g^2_{\rm YM}(\mu)} = \frac{1}{g^2_{\rm YM}(\Lambda)} \mp \frac{1}{3\pi^2}\log\frac{\mu^2}{\Lambda^2} \quad . \tag{332}$$

Finally from Eq. (332) one reads the expected β-function [30]

$$\beta(g_{YM}) = \pm \frac{g^3_{YM}}{(4\pi)^2}\frac{16}{3} \quad . \tag{333}$$

As already observed, in the planar limit $N \to \infty$ with $\lambda = g^2_{\rm YM}N$ fixed, the ratio $\beta(g_{\rm YM})/g_{\rm YM}$ reduces to zero and coincides with the one of its parent theory $\mathcal{N} = 4$ SYM.

It is also interesting to write down the one-loop amplitude given by Eq. (329) in the closed string channel by performing the modular transformation $\tau = 1/4t$, as shown in A:

$$\begin{aligned} Z^c(F)_{\Omega' I_6} \quad &= \quad \pm \frac{1}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta + \hat{F})} \int_0^\infty \frac{dt}{t^3} {\rm e}^{-\frac{y^2}{8\pi\alpha' t}} \sin\pi\nu_f \sin\pi\nu_g \\ &\quad \frac{f_2^4(iq)\Theta_2\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_2\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)}{f_1^4(iq)\Theta_1\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_1\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)} \quad . \end{aligned} \tag{334}$$

Expanding this equation up to the second order in the external field gives ($q = e^{-\pi t}$):

$$\frac{1}{g^2_{\rm YM}} = \pm \frac{1}{(4\pi)^2} \int_0^\infty \frac{dt}{4t^3} e^{-\frac{y^2}{8\pi\alpha' t}} \left(\frac{f_2(iq)}{f_1(iq)}\right)^8 \left[\frac{4}{3} + \frac{1}{\pi}\partial_t \log f_2^4(iq)\right] \quad . \tag{335}$$

Under the inverse modular transformation $t = 1/4\tau$ this equation perfectly reproduces the expression obtained in the open channel (Eq. (330)), as one can easily check by using Eq. (365). The field theory limit of the previous expression, realized as $t \to \infty$ and $\alpha' \to 0$ with $s = 2\pi\alpha' t$ fixed, gives a vanishing contribution . One could be led to conclude that the gauge/gravity correspondence does not work in this non-supersymmetric model. However, in the planar limit ($N \to \infty$ and $g^2_{\mathrm{YM}} N$ fixed) the theory has a vanishing β-function, as already noticed after Eq. (333). Therefore one can conclude that in the large N limit, where the Möbius strip contribution is suppressed and the gauge theory recovers the Bose-Fermi degeneracy in its spectrum, the gauge/gravity correspondence holds and admits a consistent interpretation in terms of open/closed string duality.

7.4 Orbifold $\mathbb{C}^2/\mathbb{Z}_2$

The gauge theory living on the fractional D3 branes is the $\mathbb{Z}_2$ invariant subsector of the open string spectrum introduced in Sect. 7.1. By using Eq.s (17) and (142) it is easy to see that the spectrum contains one $SU(N)$ gauge field, two real scalars in the adjoint representation of the gauge group and two Dirac fermions in the two-index symmetric (or antisymmetric) representation. Notice that the spectrum has a *common sector* [34] with $\mathcal{N} = 2$ SYM, namely the bosonic one. However, because of the fermionic contributions which are different, the one-loop β-function of our theory contains a subleading correction in $1/N$ with respect to $\mathcal{N} = 2$ β-function:

$$\beta(g_{YM}) = \frac{g^3_{YM}}{(4\pi)^2}\left[-\frac{11}{3}N + 2\frac{N}{6} + 2\frac{4}{3}\frac{N \pm 2}{2}\right] = \frac{g^3_{YM} N}{(4\pi)^2}\left[-2 \pm \frac{8}{3N}\right] . \tag{336}$$

In the large N limit the subleading term in $1/N$ is suppressed and the two β-functions coincide. This circumstance signals the existence of a planar equivalence between the two theories at one-loop and suggests the possibility of an extension of such equivalence at all perturbative orders. The one-loop vacuum amplitude of an open string stretching between

Table 2: Spectrum of the $SU(N)$ world-volume gauge theory of N D3 branes on the top of the O3-plane.

	$0B/\Omega' I_6 (-1)^{F_L}$ *on flat space*	$0B/\Omega' I_6 (-1)^{F_L}$ *on* $\mathbb{C}^2/\mathbb{Z}_2$	$0B/\Omega' I_6 (-1)^{F_L}$ *on* $\mathbb{C}^3/(\mathbb{Z}_2{\times}\mathbb{Z}_2)$
Gauge vectors	Adj ($2N^2$ d.o.f.)	Adj ($2N^2$ d.o.f.)	Adj ($2N^2$ d.o.f.)
Scalars	$6\times$ Adj ($6N^2$ d.o.f.)	$2\times$ Adj ($2N^2$ d.o.f.)	–
Dirac fermions	$4\times\square\!\square$ or $4\times\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ ($8N^2 \pm 8N$ d.o.f.)	$2\times\square\!\square$ or $2\times\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ ($4N^2 \pm 4N$ d.o.f.)	$\square\!\square$ or $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ ($2N^2 \pm 2N$ d.o.f.)
1-loop β-function	$\pm\frac{g^3_{YM}}{16\pi^2}\frac{16}{3}$	$\frac{N g^3_{YM}}{16\pi^2}\left(-2\pm\frac{8}{3N}\right)$	$\frac{N g^3_{YM}}{16\pi^2}\left(-3\pm\frac{4}{3N}\right)$
Parent theory	$\mathcal{N} = 4$ SYM	$\mathcal{N} = 2$ SYM	$\mathcal{N} = 1$ SYM

a stack of N undressed D3 branes and a dressed one is given by:

$$Z = \int_0^\infty \frac{d\tau}{\tau} Tr_{\text{NS}-\text{R}} \left[\left(\frac{1+h}{2} \right) \left(\frac{e+\Omega' I_6}{2} \right) P_{(-1)^{F_s}} \right.$$

$$\left. \times (-1)^{G_{bc}} P_{GSO} e^{-2\pi\tau L_0} \right] \equiv Z_e^o + Z_{\Omega' I_6}^o + Z_{he}^o + Z_{h\Omega' I_6}^o \quad , \tag{337}$$

where the trace over the Chan-Paton factors has been understood and we have used the following notation:

$$\begin{aligned} Z_e^o &\equiv \frac{1}{2}\left(Z_e^o + Z_{e(-1)^{F_s}}^o \right) \qquad & Z_{he}^o \equiv \frac{1}{2}\left(Z_{he}^o + Z_{he(-1)^{F_s}}^o \right) \\ Z_{\Omega' I_6}^o &\equiv \frac{1}{2}\left(Z_{\Omega' I_6}^o + Z_{\Omega' I_6 (-1)^{F_s}}^o \right) \qquad & Z_{h\Omega' I_6}^o \equiv \frac{1}{2}\left(Z_{h\Omega' I_6}^o + Z_{h\Omega' I_6(-1)^{F_s}}^o \right) . \end{aligned} \tag{338}$$

However notice that the term $(-1)^{F_s}$ gives a non vanishing contribution to the trace on the Chan-Paton factors, only if it appears together with the projector $\Omega' I_6$, namely in the terms in the second line of Eq. (338).

The first two terms of Eq. (337) are the ones we have already computed in the previous section (apart from an additional factor $\frac{1}{2}$ coming from the orbifold projection). Here we need just to evaluate the last two terms. The third term turns out to be equal to the one appearing in the pure orbifold calculation given in Eq. (41), as follows from the fact that

$$\text{Tr}\langle ij|e|nm\rangle = \delta_{jj}\delta_{mm} = 4N \qquad \text{Tr}\langle ij|(-1)^{F_S}|nm\rangle = 0 \quad , \tag{339}$$

where the indices $i, m = 1, ..., N, N+3, ..., 2N+2$ enumerate respectively the stack of N undressed branes and their images, while the indices $j, n = N+1, N+2$ indicate the dressed brane and its image.

Analogously, the last term in Eq. (337) can be obtained from Eq. (41) with the substitution $k \to ik$. As noticed after Eq. (2.5) of Ref. [17], in the twisted sector, only the NS and NS$(-1)^F$ and R$(-1)^F$ sectors contribute to the amplitude. However, the presence of the type 0B projector $\frac{1+(-1)^{F_s}}{2}$ makes the NS and NS$(-1)^F$ contributions to vanish because of Eq.s (318) and (319). Thus the only twisted sector which gives a non vanishing contribution to the interaction is the $R(-1)^F$ which is equal to

$$Z_{h\Omega' I_6}^o = \mp \frac{2i}{32\pi^2} \int d^4x \, F_{\alpha\beta}^a \tilde{F}^{a\,\alpha\beta} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \quad . \tag{340}$$

The overall factor 2 again takes into account the two inequivalent configurations that we have to consider in evaluating the trace, as discussed after Eq. (329).

Then the coefficient of the gauge kinetic term in this theory comes only from the second and third term of Eq. (337) corresponding to the untwisted Möbius strip and twisted annulus and turns out to be:

$$\frac{1}{g_{YM}^2} = -\frac{1}{16\pi^2} \int_0^\infty \frac{d\tau}{\tau} e^{-\frac{y^2\tau}{2\pi\alpha'}} \left\{ \left[2N \mp \frac{8}{3} \right] + \boldsymbol{\Delta} \right\} \quad , \tag{341}$$

where the first two terms give the massless states contributions to the running coupling constant coming from the twisted annulus (leading term in N) and the untwisted Moebius strip (subleading term in N). The last term, given by:

$$\mathbf{\Delta} = \pm\frac{1}{2}\left[\frac{16}{3} - \left(\frac{f_2(ik)}{f_1(ik)}\right)^8 \left(\frac{1}{3\tau^2} + k\partial_k \log f_2^4(ik)\right)\right] \quad , \tag{342}$$

contains the threshold correction due to the massive string states, which are subleading in N because they come entirely from the untwisted Moebius strip.

In the field theory limit only the massless states contributions survive and one gets:

$$\frac{1}{g_{YM}^2} = \frac{1}{16\pi^2}\left[2N \mp \frac{8}{3}\right] \log\frac{\mu^2}{\Lambda^2} \quad , \tag{343}$$

consistently with our previous calculation in Eq. (336). Moreover from Eq.s (41) and (340), following the same procedure as in Ref. [17], one can read also the θ angle which receive contributions only from the massless states propagating in the twisted annulus and Moebius strip and turns out to be

$$\theta_{YM} = -2\theta(N \pm 2) \quad , \tag{344}$$

where θ is the phase of the complex cut-off $\Lambda e^{-i\theta}$. Notice that, differently from what happens for the running coupling constant, neither the leading nor the subleading term in N are affected by threshold corrections.

The gauge theory so obtained shares some common features with $\mathcal{N} = 2$ SYM. As previously noticed, the running coupling constant and the β-function of this theory, in the large N limit, reproduce those of $\mathcal{N} = 2$ SYM. Moreover also the θ_{YM} angle in Eq. (61) in the large N limit reduces to the one of $\mathcal{N} = 2$ SYM, implying that, in the planar limit, the two theories are very close to each other. This connection appears as the natural extension to the case $\mathcal{N} = 2$ of the Armoni, Shifman and Veneziano planar equivalence for $\mathcal{N} = 1$, in which the *parent* theory is the $\mathcal{N} = 2$ SYM and the *daughter* theory is the world-volume theory of N fractional branes of our orbifold.

Finally, it is useful to rewrite the previous expressions in the closed string channel. By using Eq.s (380) and (390) and the well-known modular transformation properties of the Θ-functions, we can rewrite Eq.s (41), (340) and $Z^o_{\Omega' I_6}$ in the closed string channel. The other terms in the free energy are irrelevant in the forthcoming discussion because they are vanishing in the field theory limit. From the annulus we obtain:

$$\begin{aligned} Z^c_{he} \;=\;& \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{dt}{t}\mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}}\left[\frac{4\sin\pi\nu_f\sin\pi\nu_g}{\Theta_4^2(0|it)\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)}\right] \\ & \left[\Theta_2^2(0|it)\Theta_3(\nu_f|it)\Theta_3(\nu_g|it) - \Theta_3^2(0|it)\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)\right] \\ & -\frac{iN}{32\pi^2}\int d^4x\, F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty \frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} \quad , \end{aligned} \tag{345}$$

while from the Möbius strip:

$$
\begin{aligned}
Z^c_{\Omega' I_6}(F) &= \mp \frac{1}{(8\pi^2\alpha')^2}\int d^4x \sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{dt}{4t^3}\mathrm{e}^{-\frac{y^2}{8\pi\alpha' t}}\sin\pi\nu_f \sin\pi\nu_g \\
& \quad \frac{f_2^4(iq)\Theta_2\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_2\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)}{f_1^4(iq)\Theta_1\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)\Theta_1\left(\frac{\nu_g}{2}|it+\frac{1}{2}\right)} && (346) \\
Z^c_{h\Omega' I_6}(F) &= \mp \frac{2i}{32\pi^2}\int d^4x\, F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty \frac{dt}{t}e^{-\frac{y^2}{8\pi\alpha' t}} \quad . && (347)
\end{aligned}
$$

By expanding the previous expressions up to quadratic terms in the external field and isolating only the terms depending on the gauge field, we have from the annulus:

$$
\begin{aligned}
Z^c_h(F) &\rightarrow \left[-\frac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right]\left\{-\frac{N}{8\pi^2}\int_0^\infty \frac{dt}{t}\mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}}\right\} \\
&- iN\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\right]\int_0^\infty \frac{dt}{t}\mathrm{e}^{-\frac{y^2}{2\pi\alpha' t}} \quad . && (348)
\end{aligned}
$$

The last two equations are exact at string level even if they receive a contribution only from the massless closed string states. For $y=0$ both of them are left invariant under open/closed string duality and for this reason one expects to obtain, from the closed channel, the planar contribution to the β-function and the complete expression of the θ-angle.

By expanding the Moebius strip amplitude to the second order in the background field, one gets:

$$
\begin{aligned}
& Z^c_{\Omega' I_6}(F) \rightarrow \left[-\tfrac{1}{4}\int d^4x F^a_{\alpha\beta}F^{a\,\alpha\beta}\right] \\
&\times\left\{\pm\tfrac{1}{(4\pi)^2}\int_0^\infty \tfrac{dt}{8t^3}e^{-\frac{y^2}{8\pi\alpha' t}}\left(\tfrac{f_2(iq)}{f_1(iq)}\right)^8\left[\tfrac{4}{3}+\tfrac{1}{\pi}\partial_t\log f_2^4(iq)\right]\right\} \quad . && (349)
\end{aligned}
$$

In this case the massless pole in the open channel is not left invariant under open/closed string duality and by performing the field theory limit on such expression, we obtain a vanishing result.

Summarizing, from Eq.s (348) and (349) we get the following expression for the running coupling constant at the full closed string level:

$$
\begin{aligned}
\frac{1}{g^2_{YM}} &= -\frac{N}{8\pi^2}\int_0^\infty \frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} \\
&\pm\frac{1}{16\pi^2}\int_0^\infty \frac{dt}{8^3}e^{-\frac{y^2}{8\pi\alpha' t}}\left(\frac{f_2(iq)}{f_1(iq)}\right)^8\left[\frac{4}{3}+\frac{1}{\pi}\partial_t\log f_2^4(iq)\right] \quad , && (350)
\end{aligned}
$$

where the first line corresponds to the massless states contribution coming from the twisted cylinder Z^c_h, while the second line corresponds to the threshold corrections coming from $Z^c_{\Omega' I_6}$ which are subleading in N. By performing the field theory limit, only the first line survives and therefore one can conclude that the closed channel is able to reproduce only the contribution of the planar diagrams to the β function in Eq. (336), but not also the contribution of the non planar ones.

In conclusion, we can assert that the gauge/gravity correspondence certainly holds in the planar limit. However, some non planar information can be still obtained from the closed channel as the example of the θ-angle has showed.

7.5 Orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$

Let us consider now the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ of our orientifold theory. The action of this orbifold has been discussed in Section 3.

The states left invariant are one gauge vector and one Dirac fermion in the two-index symmetric (or anti-symmetric) representation of the gauge group. Also in this case the theory has a common bosonic sector with a supersymmetric model, that is $\mathcal{N}=1$ SYM. It is simple to check that the β-function for this theory is, at one-loop:

$$\beta(g_{YM}) = \frac{g_{YM}^3}{(4\pi)^2}\left[-\frac{11}{3}N + \frac{4}{3}\frac{N\pm 2}{2}\right] = \frac{g_{YM}^3 N}{(4\pi)^2}\left[-3 \pm \frac{4}{3N}\right] \quad , \tag{351}$$

which differs from the one of $\mathcal{N}=1$ SYM because of the subleading term in $1/N$.

Notice that for $N=3$ the two-index antisymmetric representation is equal to the fundamental one. Therefore, with the antisymmetric choice, *the world-volume gauge theory living on a stack of N fractional branes in the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ of the orientifold type $0B/\,\Omega' I_6(-1)^{F_L}$, for $N=3$ is nothing but one flavour QCD*. This is an alternative and simpler stringy realization of the Armoni-Shifman-Veneziano model. In Ref. [34] and references therein, the same gauge theory is realized, in the framework of type 0A theory, by considering a stack of N D4 branes on top of an orientifold O4-plane, suspended between orthogonal NS 5 branes. It would be interesting to exploit the relation between the two models which should be connected by a simple T-duality.

Besides the stringy realization, the gauge theory we end up with is related by planar equivalence to $\mathcal{N}=1$ SYM. In the language of Armoni, Shifmann and Veneziano, the symmetric $(+)$ and antisymmetric $(-)$ choices correspond to the S and A orientifold theories of $\mathcal{N}=1$ SYM. This opens the way to a very interesting extension of many predictions of supersymmetric parent theory to the non-supersymmetric daughter theory, which holds in the large N limit.[34]

As discussed in Sect. 3.3, the orbifold $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ can be seen as obtained by three copies of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$ where the i-th $\mathbb{Z}_2$ contains the elements $(1,h_i)$ $(i=1,\dots 3)$.

In particular we consider the one-loop vacuum amplitude of an open string stretching between a stack of N_I $(I=1,\dots,4)$ branes of type I and a D3-fractional brane of type $I=1$ dressed with an $SU(N)$ gauge field. In this case the amplitude turns out to be given by the sum of eight terms:

$$Z = \int_0^\infty \frac{d\tau}{\tau} Tr_{\mathrm{NS-R}}\left[\left(\frac{1+h_1+h_2+h_3}{4}\right)\left(\frac{e+\Omega' I_6}{2}\right) P_{(-1)^{F_s}}\right.$$

$$\left.\times\,(-1)^{G_{bc}} P_{GSO} e^{-2\pi\tau L_0}\right] \equiv Z_e^o + Z_{\Omega' I_6}^o + \sum_{i=1}^{3}\left[Z_{h_i e}^o + Z_{h_i\Omega' I_6}^o\right] \quad . \tag{352}$$

Here the first two terms are the same as the ones of the previous orbifolds except for a

further factor 1/2 due to the orbifold projection. The terms $Z^o_{h_ie}$ turn out to be

$$
\begin{aligned}
Z^o_{h_i} \quad = \quad & \frac{f_i(N)}{2\,(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}}\frac{4\,\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \\
& \times\left\{\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau)-\Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right\} \\
& -\frac{i\,f_i(N)}{64\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty\frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}} \quad , \qquad (353)
\end{aligned}
$$

where the functions $f_i(N)$ are given in Eq. (83). As in the previous orbifold case, all the bosonic terms of $Z_{h_i\Omega' I_6}$ vanish because of the contribution to the trace of the projector $P_{(-1)^{F_s}}$, while the $R(-1)^F$ sector gives

$$
Z^o_{h_i\Omega' I_6} = \mp\frac{2i}{64\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\,\alpha\beta}\int_0^\infty\frac{d\tau}{\tau}\mathrm{e}^{-\frac{y^2\tau}{2\pi\alpha'}} \quad . \qquad (354)
$$

By extracting the coefficient of the gauge kinetic term from the field theory limit of the amplitude in Eq. (352) and specializing to the case $N_1 = N$, $N_2 = N_3 = N_4 = 0$ we get:

$$
\frac{1}{g^2_{YM}} = \frac{1}{16\pi^2}\left[3N\mp\frac{4}{3}\right]\log\frac{\mu^2}{\Lambda^2} \quad , \qquad (355)
$$

while the theta angle θ_{YM} turns out to be

$$
\theta_{\mathrm{YM}} = -(N\pm 2)\theta \quad . \qquad (356)
$$

We can repeat the same analysis in the closed string channel by transforming under open/closed string duality Eq. (352) and performing all the steps explained in the last subsection. However, being Eq.s (82), (354) and $Z^o_{\Omega' I_6}$ coincident, apart from an overall factor, with Eq.s (41) and (340) we get the same conclusions as we did in that subsection. In the closed string channel one is able to capture, in the field theory limit, only the planar contribution to the β-function and the complete expression for the θ-angle. Gauge/gravity correspondence in these non-supersymmetric models has a full consistent interpretation in terms of open/closed string duality only in the large N limit even if some non planar results are still present in the closed channel.

8 Conclusion

In this article we have investigated the conditions that have to be satisfied for making the gauge/gravity correspondence to be at work, both in supersymmetric and non-supersymmetric string theories. The supersymmetric theories, that we have examined, are $\mathcal{N}$=1,2 SQCD, showing that the gauge/gravity correspondence is a consequence of the open/closed string duality *if* the threshold corrections, i.e. the contribution of the massive string states to the gauge coupling constant and θ_{YM}-angle, vanish. Indeed when this happens the contribution of the massless open string states is necessarily mapped, under open/closed string duality, into the one of the massless closed string states and this provides the reason of why it is possible to get gauge-theories quantities from supergravity.

This equivalence between massless states in the two channels has an interesting consequence. In fact, on the one hand, it is clear from Eq. (51) that the massless open string states generate an UV divergence for small values of the modular parameter τ. On the other hand, open/closed string duality together with the absence of threshold corrections, transforms such a behaviour in an IR logarithmic divergence in the closed string channel. Such a divergence in the closed string channel has a precise physical meaning: it corresponds to the propagation in the two transverse dimensions not included among those on which the orbifold acts, of some bulk field. We can therefore conclude that the vanishing of the threshold corrections determines the propagation in two dimensions of some bulk fields which contribute, through the holographic identities, to the gauge-theory parameters. This is exactly what happens in the supersymmetric models taken in consideration where those bulk fields are in fact the twisted fields.

The same analysis has then been performed for non-supersymmetric models. In particular, we have considered D3 branes stuck at the fixed point of orientifolds of type 0 string theory. The gauge theory living on the world-volume of such branes provides an example of the so-called *orientifold field theories*. The most interesting model has been obtained by taking fractional D3 branes in type $0/\Omega' I_6(-1)^{F_L}$ in the orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In fact in this case we get QCD with one flavour. In these models we have shown that, for the θ_{YM}-angle, the threshold corrections are absent, while, for the gauge coupling constant, they are vanishing only in the large N limit, As a consequence, the θ-angle can be exactly obtained from the closed string channel, while the gauge coupling constant can be obtained from the closed string channel only for large N. Again their values are given by massless closed string fields propagating in two dimensions.

Table 3: Summary

THEORY	GAUGE/GRAVITY	BOSE/FERMI degeneracy	NO FORCE	THRESHOLD CORRECTIONS
Bosonic	No	No	No	$\neq 0$
type 0B	No	No	No	$\neq 0$
Dyonic branes in type 0B	Yes	Yes	Yes	$= 0$
0'B	Yes	Yes	Yes	$= 0$
$0B/\Omega' I_6(-1)^{F_L}$ (g_{YM})	Yes for $N \to \infty$	Yes for $N \to \infty$	Yes for $N \to \infty$	$= 0$ for $N \to \infty$
$0B/\Omega' I_6(-1)^{F_L}$ (θ_{YM})	Yes also for finite N	No for finite N	Yes for finite N	$= 0$ also for finite N

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A Θ-functions

The Θ-functions which are the solutions of the heat equation:

$$\frac{\partial}{\partial\tau}\Theta\left(\nu|i\tau\right) = \frac{1}{4\pi}\partial_\nu^2\Theta(\nu|i\tau) \tag{357}$$

are given by

$$\Theta_1(\nu|it) \equiv \Theta_{11}(\nu,|it) = -2q^{\frac{1}{4}} \sin\pi\nu \prod_{n=1}^{\infty} \left[(1-q^{2n})(1-e^{2i\pi\nu}q^{2n})(1-e^{-2i\pi\nu}q^{2n})\right]$$
$$\Theta_2(\nu|it) \equiv \Theta_{10}(\nu,|it) = 2q^{\frac{1}{4}} \cos\pi\nu \prod_{n=1}^{\infty} \left[(1-q^{2n})(1+e^{2i\pi\nu}q^{2n})(1+e^{-2i\pi\nu}q^{2n})\right]$$
$$\Theta_3(\nu,|it) \equiv \Theta_{00}(\nu,|it) = \prod_{n=1}^{\infty} \left[(1-q^{2n})(1+e^{2i\pi\nu}q^{2n-1})(1+e^{-2i\pi\nu}q^{2n-1})\right]$$
$$\Theta_4(\nu,|it) \equiv \Theta_{01}(\nu,|it) = \prod_{n=1}^{\infty} \left[(1-q^{2n})(1-e^{2i\pi\nu}q^{2n-1})(1-e^{-2i\pi\nu}q^{2n-1})\right], \qquad (358)$$

with $q = e^{-\pi\tau}$. The modular transformation properties of the Θ functions are

$$\Theta_1(\nu|it) = i\,\Theta_1(-i\frac{\nu}{t}|\frac{i}{t})e^{-\pi\nu^2/t}t^{-\frac{1}{2}}$$
$$\Theta_{2,3,4}(\nu|it) = \Theta_{4,3,2}(-i\frac{\nu}{t}|\frac{i}{t})e^{-\pi\nu^2/t}t^{-\frac{1}{2}} \quad . \qquad (359)$$

It is also useful to define the f-functions and their transformation properties:

$$f_1 \equiv q^{\frac{1}{12}} \prod_{n=1}^{\infty}(1-q^{2n}) \quad ; \qquad (360)$$

$$f_2 \equiv \sqrt{2}q^{\frac{1}{12}} \prod_{n=1}^{\infty}(1+q^{2n}) \quad ; \qquad (361)$$

$$f_3 \equiv q^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1+q^{2n-1}) \quad ; \qquad (362)$$

$$f_4 \equiv q^{-\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^{2n-1}) \quad . \qquad (363)$$

In the case of a real argument $q = e^{-\pi t}$ they transform as follows under the modular transformation $t \to 1/t$:

$$f_1(e^{-\frac{\pi}{t}}) = \sqrt{t}f_1(e^{-\pi t}) \quad ; \quad f_2(e^{-\frac{\pi}{t}}) = f_4(e^{-\pi t}) \quad ; f_3(e^{-\pi t}) = f_3(e^{-\frac{\pi}{t}}), \qquad (364)$$

while for complex argument one gets:[82]

$$f_1(ie^{-\pi t}) = (2t)^{-1/2} f_1(ie^{-\frac{\pi}{4t}}) \qquad f_2(ie^{-\pi t}) = f_2(ie^{-\frac{\pi}{4t}}) \qquad (365)$$
$$f_3(ie^{-\pi t}) = \quad e^{i\pi/8} f_4(ie^{-\frac{\pi}{4t}}) \qquad f_4(ie^{-\pi t}) = e^{-i\pi/8} f_3(ie^{-\frac{\pi}{4t}}) \; . \qquad (366)$$

The following relations are also useful:

$$\Theta_{2,3,4}(0|it) = f_1(e^{-\pi t})\, f_{2,3,4}^2(e^{-\pi t}) \quad ; \quad \lim_{\nu\to 0} \frac{\Theta_1(\nu|it)}{2\sin\pi\nu} = -f_1^3(e^{-\pi t}) \; . \qquad (367)$$

It is also useful to give an alternative representation of the Θ-functions:

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|t) = \sum_{n=-\infty}^{\infty} e^{2\pi i\left[\frac{1}{2}(n+\frac{a}{2})^2 t+(n+\frac{a}{2})(\nu+\frac{b}{2})\right]}, \tag{368}$$

where a, b are rational numbers. It is easy to show that

$$\Theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\nu|t) = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi t(n-\frac{1}{2})^2} e^{i\pi\nu(2n-1)} \equiv \Theta_1(\nu|t) \tag{369}$$

and

$$\Theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\nu|t) = \sum_{n=-\infty}^{\infty} e^{i\pi t(n-\frac{1}{2})^2} e^{i\pi\nu(2n-1)} = \Theta_2(\nu|t) \equiv -\Theta_1\left(\nu + \frac{1}{2}|t\right). \tag{370}$$

From the definition in Eq. (368) it is easy to derive the following identities:

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} \left(\nu + \frac{\epsilon_1}{2}t + \frac{\epsilon_2}{2}|t\right) = e^{-\frac{i\pi t\epsilon_1^2}{4}} e^{-\frac{i\pi\epsilon_1}{2}(2\nu+b)} e^{-\frac{i\pi\epsilon_1\epsilon_2}{2}} \Theta \begin{bmatrix} a+\epsilon_1 \\ b+\epsilon_2 \end{bmatrix} (\nu|t). \tag{371}$$

and

$$\frac{1}{2}\sum_{a,b=0}^{1} (-1)^{a+b+ab} \prod_{i=1}^{4} \Theta \begin{bmatrix} a+h_i \\ b+g_i \end{bmatrix} (\nu_i) = -\prod_{i=1}^{4} \Theta \begin{bmatrix} 1-h_i \\ 1-g_i \end{bmatrix} (\nu_i') \quad , \tag{372}$$

with $\sum_i h_i = \sum_i g_i = 0$ and $\nu_i' \equiv \frac{1}{2}\left(-\nu_i + \sum_{j\neq i}\nu_j\right)$. Eq. (371) can be used to write a different expression for the Θ-function. In fact, by applying such equation with $a = b = 0$, we get:

$$\Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left(\nu + \frac{\epsilon_1}{2}t + \frac{\epsilon_2}{2}|t\right) = e^{-\frac{i\pi t\epsilon_1^2}{4}} e^{-i\pi\epsilon_1\nu} e^{-\frac{i\pi\epsilon_1\epsilon_2}{2}} \Theta \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} (\nu|t) \ . \tag{373}$$

Redefining $\epsilon_1 \equiv a$ and $\epsilon_2 \equiv b$, we have:

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\nu|i\,t) = e^{-\frac{\pi t a^2}{4}} e^{i\pi b(\nu+\frac{a}{2})} \prod_{n=1}^{+\infty} (1-q^{2n})\left(1+e^{2i\pi(\nu+\frac{b}{2})}q^{2n-1+a}\right) \times \left(1+e^{-2i\pi(\nu+\frac{b}{2})}q^{2n-1-a}\right) \quad . \tag{374}$$

Under an arbitrary modular transformation $t \to \frac{at+b}{ct+d}$, Θ_1 transforms as follows

$$\Theta_1\left(\frac{\nu}{ct+d}|\frac{at+b}{ct+d}\right) = \eta'\,\Theta_1(\nu|t)e^{i\pi c\nu^2/(ct+d)}(ct+d)^{\frac{1}{2}} \quad , \tag{375}$$

where η' is an eighth-root of unity. It implies the following transformation of Θ_1:

$$\Theta_1\left(-\frac{\nu}{2}|\frac{t}{4}-\frac{1}{2}\right) = \frac{1}{\eta'}\Theta_1\left(-\frac{\nu}{t}|\frac{1}{2}-\frac{1}{t}\right) e^{-i\pi\nu^2/t}\left(\frac{2}{t}\right)^{\frac{1}{2}} \quad , \tag{376}$$

that is obtained from Eq. (376) by first making the substitutions $\nu \to -\frac{\nu}{2}$ and $t \to \frac{t}{4} - \frac{1}{2}$ and then choosing $a = d = 1, c = 2$ and $b = 0$. By performing in the previous equation the substitution $t \to 4it$ we get the following equation:

$$\Theta_1\left(-\frac{\nu}{2}|it-\frac{1}{2}\right) = \frac{1}{\eta'}\Theta_1\left(\frac{i\nu}{4t}|\frac{1}{2}+\frac{i}{4t}\right) e^{-\pi\nu^2/(4t)}\left(\frac{1}{2it}\right)^{\frac{1}{2}} \quad . \tag{377}$$

Finally, by using Eq. (375) with $a = b = d = 1$ and $c = 0$ we can write:

$$\Theta_1 (\nu|t+1) = \eta' \Theta_1 (\nu|t) \ . \tag{378}$$

This latter equation allows us to write

$$\Theta_1 \left(-\frac{\nu}{2}|it - \frac{1}{2} \right) = \frac{1}{\eta'} \Theta_1 \left(-\frac{\nu}{2}|it + \frac{1}{2} \right) \tag{379}$$

that, inserted in Eq. (377), leads to:

$$\Theta_1 \left(-\frac{\nu}{2}|it + \frac{1}{2} \right) = \Theta_1 \left(\frac{i\nu}{4t}|\frac{1}{2} + \frac{i}{4t} \right) e^{-\pi\nu^2/(4t)} \left(\frac{1}{2it} \right)^{\frac{1}{2}} \ . \tag{380}$$

In order to get the analogous transformation property of Θ_2, we use the following relation:

$$\Theta_2 \left(-\frac{\nu}{2}|\frac{t}{4} - \frac{1}{2} \right) = -\Theta_1 \left(\frac{1-\nu}{2}|\frac{t}{4} - \frac{1}{2} \right) \ . \tag{381}$$

Then, by applying Eq. (375) with $\nu \to \frac{1-\nu}{2}$, $t \to \frac{t}{4} - \frac{1}{2}$ and $a = d = 1$, $c = 2$, $b = 0$, to the second term of the previous equation, it can be rewritten as

$$\Theta_1 \left(\frac{1-\nu}{2}|\frac{t}{4} - \frac{1}{2} \right) = \frac{1}{\eta'} \left(\frac{2}{t} \right)^{1/2} e^{-\frac{i\pi(1-\nu)^2}{t}} \Theta_1 \left(\frac{1-\nu}{t}|\frac{1}{2} - \frac{1}{t} \right) \tag{382}$$

and then, by substituting it in Eq.(381) and sending $t \to 1/t$, one gets:

$$\Theta_2 \left(-\frac{\nu}{2}|\frac{1}{4t} - \frac{1}{2} \right) = -\frac{1}{\eta'} (2t)^{1/2} e^{-i\pi(1-\nu)^2 t} \Theta_1 \left((1-\nu)t\,|\frac{1}{2} - t \right) \ . \tag{383}$$

Let us consider the Θ_1 in the second term of the previous equation. By defining in it $t' \equiv \frac{1}{2} - t$ and then $\nu' \equiv -\nu \left(\frac{1}{2} - t' \right)$ we can rewrite it as

$$\Theta_1 \left((1-\nu)t\,|\frac{1}{2} - t \right) = \Theta_1 \left(\nu' - t' + \frac{1}{2}\,|t' \right) \ . \tag{384}$$

Therefore, by using Eq. (371) with $a = b = 1$, $\epsilon_2 = 1, \epsilon_1 = -2$, we can write Eq.(384) as follows:

$$\Theta_1 \left((1-\nu)t\,|\frac{1}{2} - t \right) = ie^{i\pi(1-2\nu)t} \Theta_2 \left(-\nu t\,|\frac{1}{2} - t \right) \ , \tag{385}$$

where we have restored the variables ν and t and used the following identity:

$$\Theta \begin{bmatrix} -1 \\ 2 \end{bmatrix} (\nu|t) = -\Theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\nu|t) = -\Theta_2(\nu|t) \ , \tag{386}$$

that can be easily derived starting from the general expression of the Θ-function given in Eq. (368). Then by inserting Eq. (385) in Eq. (383) we get

$$\Theta_2 \left(-\frac{\nu}{2}|\frac{1}{4t} - \frac{1}{2} \right) = -\frac{i}{\eta'} \Theta_2 \left(-\nu t|\frac{1}{2} - t \right) e^{-i\pi\nu^2 t} (2t)^{\frac{1}{2}} \ . \tag{387}$$

Furthermore, by performing the substitution $t \to -\frac{i}{4t}$, Eq. (387) becomes

$$\Theta_2\left(-\frac{\nu}{2}|it-\frac{1}{2}\right) = -\frac{i}{\eta'}\,\Theta_2\left(\frac{i\nu}{4t}|\frac{1}{2}+\frac{i}{4t}\right) e^{-\pi\nu^2/(4t)}(2it)^{-\frac{1}{2}} \quad . \tag{388}$$

Finally, by rewriting Θ_2-function in terms of Θ_1 by means of Eq. (370) and using Eq. (378), we can write:

$$\Theta_2\left(-\frac{\nu}{2}|it-\frac{1}{2}\right) = \frac{1}{\eta'}\Theta_2\left(-\frac{\nu}{2}|it+\frac{1}{2}\right) \quad . \tag{389}$$

The last identity allows us to write:

$$\Theta_2\left(-\frac{\nu}{2}|it+\frac{1}{2}\right) = -i\Theta_2\left(\frac{i\nu}{4t}|\frac{1}{2}+\frac{i}{4t}\right) e^{-\pi\nu^2/(4t)}(2it)^{-\frac{1}{2}} \quad . \tag{390}$$

B Derivation of some results

In this Appendix we explicitly derive many equations of the previous sections.

In order to derive the coefficient of the gauge kinetic term in the open string channel, we need the following expansions of the Θ-functions up to the quadratic order in the gauge fields:

$$\begin{aligned}
\Theta_n\left[i\nu_f\tau|i\tau\right] &\simeq \Theta_n\left[0|i\tau\right] + 2\frac{\tau^2}{\pi}\partial_\tau\Theta_n\left[0|i\tau\right] f^2 \\
&= f_1(k)f_n^2(k)\left[1-2\tau^2 k\frac{\partial}{\partial k}\log[f_1(k)f_n^2(k)]\, f^2\right]
\end{aligned} \tag{391}$$

for $n = 2, 3, 4$ and

$$\frac{\sin\pi\nu_f}{\Theta_1\left[i\nu_f\tau|i\tau\right]} \simeq \frac{i}{2\tau f_1^3(k)}\left[1+\left(\frac{1}{6}+\tau^2 k\frac{\partial}{\partial k}\log f_1^2(k)\right) f^2 \ldots\right] \tag{392}$$

$$\Theta_1^2(i\frac{\tau}{2}(\nu_f-\nu_g)|i\tau) \simeq -\tau^2(if-g)^2 f_1^6(e^{-\pi\tau}) \tag{393}$$

for Θ_1, together with

$$\sqrt{-\det(\eta+2\pi\alpha' F)} = 1-\frac{1}{2}(f^2-g^2)+\ldots \quad ; \quad f^2-g^2 = -\frac{(2\pi\alpha')^2}{4}F^2 \tag{394}$$

and

$$\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} = \sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} \quad . \tag{395}$$

Eqs. (391) and (392) may be proved by using the following relations:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{k^{2n-1}}{(1-k^{2n-1})^2} &= -k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^{\infty}(1-k^{2n}) + \log\prod_{n=1}^{\infty}(1-k^{2n-1})\right] \; , \\
\sum_{n=1}^{\infty}\frac{k^{2n-1}}{(1+k^{2n-1})^2} &= k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^{\infty}(1-k^{2n}) + \log\prod_{n=1}^{\infty}(1+k^{2n-1})\right] \; , \\
\sum_{n=1}^{\infty}\frac{k^{2n}}{(1+k^{2n})^2} &= k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^{\infty}(1-k^{2n}) + \log\prod_{n=1}^{\infty}(1+k^{2n})\right] \; , \\
\sum_{n=1}^{\infty}\frac{k^{2n}}{(1-k^{2n})^2} &= -k\frac{d}{dk}\left[\frac{1}{2}\log\prod_{n=1}^{\infty}(1-k^{2n})\right] \; .
\end{aligned} \tag{396}$$

For selecting the coefficient of the gauge-kinetic term in the closed channel one needs the following expansions of the Θ-functions at the presence of an external field

$$\begin{aligned}\Theta_n(\nu|it) &= \Theta_n(0|it) - \frac{2}{\pi}\partial_\tau\Theta_n(0|it)\ f^2 \qquad n=2,3,4 \\ &= f_1(e^{-\pi t})f_n^2(e^{-\pi t})\left[1-\frac{2}{\pi}\,f^2\partial_\tau\log\left(f_1(e^{-\pi t})f_n^2(e^{-\pi t})\right)\right] \end{aligned} \tag{397}$$

and

$$\frac{\sin\pi\nu_f}{\Theta_1(\nu_f|it)} \simeq -\frac{1}{2f_1^3(q)}\left\{1-2f^2q\partial_q\log\prod_n\left(1-q^{2n}\right)\right\} \tag{398}$$

for Θ_1, which has been used to obtain Eq. (100).

The Euler-Heisenberg action in Eq. (456) is obtained through the use of the following expressions which hold for $\tau\to\infty$ and $\alpha'\to 0$:

$$\Theta_1(i\nu\tau|i\tau+\frac{1}{2}) \to -2i(ik)^{1/4}\sinh\pi\nu\tau \quad , \quad \Theta_2(i\nu\tau|i\tau+\frac{1}{2}) \to 2(ik)^{1/4}\cosh\pi\nu\tau \tag{399}$$

$$\nu_f \to -2\alpha' i\hat{f} \quad , \quad \nu_g \to -2\alpha'\hat{g} \tag{400}$$

and

$$f_1(ik)\to(ik)^{1/12} \quad , \quad f_2(ik)\to\sqrt{2}(ik)^{1/12} \tag{401}$$

which, together with

$$\sqrt{-\det(\eta+\hat{F})}\sin\pi\nu_f\sin\pi\nu_g = i(2\pi\alpha')^2\hat{f}\hat{g} \quad , \tag{402}$$

lead to Eq. (456).

In order to derive Eq. (330) we need to use the expansions of the Θ-functions given in Eqs. (391) and (392) in which the second argument is $i\tau+\frac{1}{2}$ instead of $i\tau$. The effect of the previous shift is simply to change the argument of the f_i-functions in Eqs. (391) and (392) from k to ik. By inserting these equations with an imaginary argument in Eq. (329), we get ($k=e^{-\pi\tau}$):

$$\begin{aligned} Z^F &\simeq \pm 4\frac{1}{(8\pi^2\alpha')^2}\int d^4x\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[1-\frac{1}{2}(f^2-g^2)\right]\left[\frac{i}{2\tau f_1^3(ik)}\right]^2\left[\frac{f_2^8(ik)}{f_1^2(ik)}\right] \\ &\times\left[1+\left(\frac{1}{6}+\tau^2k\frac{\partial}{\partial k}\log f_1^2(ik)\right)(f^2-g^2)\right]\left[1+2(f^2-g^2)\frac{\tau^2}{\pi}\partial_\tau\log\left(f_1(ik)f_2^2(ik)\right)\right] \\ &= \mp\frac{1}{(8\pi^2\alpha')^2}\int d^4x\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left(\frac{f_2(ik)}{f_1(ik)}\right)^8 \\ &\times\left[\frac{1}{\tau^2}+\left(-\frac{1}{3\tau^2}+\frac{2}{\pi}\partial_\tau\log f_2^2(ik)\right)(f^2-g^2)\right]\ . \end{aligned} \tag{403}$$

From it we can easily obtain Eq. (330) if $f(g)=2\pi\alpha'\hat{f}(\hat{g})$ is taken into account..

In order to write the amplitude in Eq. (329) in the closed channel we need to perform the modular transformation $\tau = 1/4t$ that gives

$$Z^c(F)_{\Omega' I_6} = \mp \frac{1}{(8\pi^2\alpha')^2} \int d^4x \sqrt{-\det(\eta+\hat{F})} \int_0^\infty \frac{dt}{t^3} \sin\pi\nu_f \sin\pi\nu_g$$
$$\frac{f_2^4(ie^{-\pi t})\Theta_2\left(i\frac{\nu_f}{4t}|\frac{i}{4t}+\frac{1}{2}\right)\Theta_2\left(i\frac{\nu_g}{4t}|\frac{i}{4t}+\frac{1}{2}\right)}{f_1^4(ie^{-\pi t})\Theta_1\left(i\frac{\nu_f}{4t}|\frac{i}{4t}+\frac{1}{2}\right)\Theta_1\left(i\frac{\nu_g}{4t}|\frac{i}{4t}+\frac{1}{2}\right)} \quad . \tag{404}$$

Eq. (404) is obtained from Eq. (329) by using Eq.s (365) and by changing variable from τ to $t = \frac{1}{4\tau}$. Finally, by using Eq.s (380) and (390) one gets Eq. (334) from Eq. (404).

Let us write now the formulas for the Θ-functions that are needed to get the previous equation. We show that this is equal to Eq. (334).

In order to obtain Eq. (335) we have used the following expansions in the external field

$$\begin{aligned}\Theta_n\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right) &\simeq \Theta_n\left[0|it+\frac{1}{2}\right] - \frac{2}{\pi}\partial_t\Theta_n\left(0|it+\frac{1}{2}\right)\frac{f^2}{4} \\ &= f_1(iq)f_n^2(iq)\left[1+2q\partial_q\log[f_1(iq)f_n^2(iq)]\,\frac{f^2}{4}\right]\end{aligned} \tag{405}$$

for $n = 2, 3, 4$, and

$$\frac{\sin\pi\nu_f}{\Theta_1\left(\frac{\nu_f}{2}|it+\frac{1}{2}\right)} \simeq -\frac{1}{f_1^3(iq)}\left\{1+f^2\left[\frac{1}{8}-q\partial_q\frac{1}{2}\log\prod_n\left(1-(iq)^{2n}\right)\right]\right\} \tag{406}$$

for Θ_1.

In the following we give some equations useful in computing the traces over zero modes in the R sector (p' odd):

$$\begin{aligned}\mathrm{Tr}_R^{z.m.}\left[\Omega' I_{9-p'}(-1)^{G^0_{\beta\gamma}}\right] &= -\mathrm{Tr}\left[\Gamma^{p'+1}\ldots\Gamma^9\Gamma^{p+1}\ldots\Gamma^9\right]\mathrm{Tr}\left[(-1)^{G^0_{\beta\gamma}}\right] \\ &= -2^4(-1)^{(9-p)/2}\delta_{p,p'} \quad ,\end{aligned} \tag{407}$$

while

$$\begin{aligned}&\mathrm{Tr}_R^{z.m.}\left[\Omega' I_{9-p'}(-1)^{F^0}\right] = -\mathrm{Tr}\left[\Gamma^{11}\Gamma^{p'+1}\ldots\Gamma^9\Gamma^{p+1}\ldots\Gamma^9\right]\mathrm{Tr}\left[\mathbb{I}_{gh.}\right] \\ &= -\lim_{x\to1}\frac{(2i)^4(-2)}{1-x^2}\mathrm{Tr}\left[\prod_{k=0}^4 x^{2N_k}\prod_{k=0}^4 N_k\prod_{i=(p'+1)/2}^4(2iN_i)\prod_{i=(p+1)/2}^4(2iN_i)\right] \\ &= -2^4\delta_{|p-p'|,8} \quad ,\end{aligned} \tag{408}$$

where we have used Eq.s 963). The previous equations have been used for deriving Eq.s (273).

In order to define properly the trace over zero modes in the R-R sector in Eq. (258) and (300), let us start from considering the trace of the identity matrix $\mathbb{I}$ in the $2^{d/2}$-dimensional spinor representation:

$$\mathrm{Tr}[\mathbb{I}] = 2^{d/2} \tag{409}$$

and observe that $\mathbb{I}$ can be considered of course as the product $C^{-1}C$, being C the charge conjugation operator. Hence we have:

$$2^{d/2} = \mathrm{Tr}[\mathbb{I}] = \sum_{A,B} \left(C^{-1}\right)^{AB} (C)_{BA} = \sum_{A,B} < A|B > (C)_{BA} \quad , \tag{410}$$

where we have used

$$< A|B >= \left(C^{-1}\right)^{AB} \quad . \tag{411}$$

This shows that:

$$\mathrm{Tr}[\mathbb{I}] = \sum_{A,B} < A|\mathbb{I}|B > C_{BA} \tag{412}$$

and therefore for any operator $\mathcal{O}$ one has:

$$\mathrm{Tr}[\mathcal{O}] = \sum_{A,B} < A|\mathcal{O}|B > C_{BA} \quad . \tag{413}$$

Let us apply this definition to $\mathrm{Tr}\left[\Omega' I_{9-p'}\right]$:

$$\mathrm{Tr}\left[\Omega' I_{9-p'}\right] = \sum_{A,B} < \tilde{C}| < D|\Omega^{'} I_{9-p'}|A > |\tilde{B} > (C)_{BC}(C)_{AD} \quad . \tag{414}$$

We know that, for p odd, the following equation holds:

$$\begin{aligned} &\Omega' I_{9-p'}\left[|A >_{-\frac{1}{2}} |\tilde{B} >_{-\frac{1}{2}}\right] \\ &= \left(\Gamma^9 \dots \Gamma^{p'+1}\right)^A_F \left(\Gamma^9 \dots \Gamma^{p'+1}\Gamma^{11}\right)^B_E |E >_{-\frac{1}{2}} |\tilde{F} >_{-\frac{1}{2}} (C)_{BC}(C)_{AD}. \end{aligned} \tag{415}$$

Hence the trace (414) becomes:

$$\begin{aligned} \mathrm{Tr}\left[\Omega' I_{9-p'}\right] &= {}_{-\frac{1}{2}}< \tilde{C}|_{-\frac{1}{2}} < D| \left(\Gamma^9 \dots \Gamma^{p'+1}\right)^A_F \left(\Gamma^9 \dots \Gamma^{p'+1}\Gamma^{11}\right)^B_E |E >_{-\frac{1}{2}} |\tilde{F} >_{-\frac{1}{2}} (C)_{BC}(C)_{AD} \\ &= (C^{-1})^{CF}(C^{-1})^{DE} \left(\Gamma^9 \dots \Gamma^{p'+1}\right)^A_F (\left(\Gamma^9 \dots \Gamma^{p'+1}\Gamma^{11}\right)^B_E (C_{BC})(C_{AD}) \\ &= \left[\Gamma^9 \cdots \Gamma^{p'+1}\right]^A_F \left[\left(C^{-1}\right)^T\right]^{FC} \left[C^T\right]_{CB} [\left(\Gamma^9 \dots \Gamma^{p'+1}\Gamma^{11}\right]^B_E \left[\left(C^{-1}\right)^T\right]^{ED} \left[C^T\right]_{DA} \\ &= \mathrm{Tr}\left[\Gamma^9 \dots \Gamma^{p'+1}(C^{-1})^T C\Gamma^9 \dots \Gamma^{p'+1}\Gamma^{11}(C^{-1})^T C\right] \\ &= \mathrm{Tr}\left[\Gamma^9 \dots \Gamma^{p'+1}\Gamma^9 \dots \Gamma^{p'+1}\Gamma^{11}\right] = 0 \, , \end{aligned} \tag{416}$$

where we have used $(C^{-1})^T = -C^{-1}$ and $\mathrm{Tr}\left[\Gamma^{11}\right] = 0$.

In the following we compute the superghost zero modes contribution to the interaction between two branes and we show that it coincides with the results known in literature.

The zero-mode part of the supeghost boundary state is given by:

$$\begin{aligned} |B_{\mathrm{sgh}}, \eta\rangle^{(0)}_{\mathrm{R-R}} &= \frac{1}{\sqrt{2}}\left[\frac{e^{i\eta\gamma_0\tilde{\beta}_0}}{1+i\eta}|0\rangle_{-\frac{1}{2}} \otimes |\tilde{0}\rangle_{-\frac{3}{2}} + \frac{e^{-i\eta\tilde{\gamma}_0\beta_0}}{1-i\eta}|\tilde{0}\rangle_{-\frac{1}{2}} \otimes |0\rangle_{-\frac{3}{2}}\right] \\ &= \frac{1}{\sqrt{2}}\left[|B^{(1)}_{\mathrm{sgh}}, \eta\rangle^{(0)}_{\mathrm{R-R}} + |B^{(2)}_{\mathrm{sgh}}, \eta\rangle^{(0)}_{\mathrm{R-R}}\right]. \end{aligned} \tag{417}$$

Consistently, we also have:

$$
\begin{aligned}
{}^{(0)}_{\mathrm{R-R}}\langle \eta, B_{\mathrm{sgh}}| &= \frac{1}{\sqrt{2}}\left[{}_{-\frac{1}{2}}\langle\tilde{0}| \otimes {}_{-\frac{3}{2}}\langle 0| \frac{e^{-i\eta\beta_0\tilde{\gamma}_0}}{1-i\eta} + {}_{-\frac{3}{2}}\langle\tilde{0}| \otimes {}_{-\frac{1}{2}}\langle 0| \frac{e^{i\eta\tilde{\beta}_0\gamma_0}}{1+i\eta} \right] \\
&= \frac{1}{\sqrt{2}}\left[{}^{(0)}_{\mathrm{R-R}}\langle \eta, B^{(1)}_{\mathrm{sgh}}| + + {}^{(0)}_{\mathrm{R-R}}\langle \eta, B^{(2)}_{\mathrm{sgh}}| \right]. \qquad (418)
\end{aligned}
$$

Let us now compute

$$
\begin{aligned}
{}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B_{\mathrm{sgh}}|B_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} &\equiv \frac{1}{2}\lim_{x\to 1}\left[{}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B^{(1)}_{\mathrm{sgh}}| x^{-2\gamma_0\beta_0} |B^{(1)}_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} \right. \\
&+ \left. {}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B^{(2)}_{\mathrm{sgh}}| x^{-2\gamma_0\beta_0} |B^{(2)}_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} \right]. \qquad (419)
\end{aligned}
$$

The first contribution to the previous expression is the one given in Ref.s [85] and [86], i.e:

$$
\begin{aligned}
{}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B^{(1)}_{\mathrm{sgh}}| x^{-2\gamma_0\beta_0} |B^{(1)}_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} &= \frac{1}{1+i(\eta_1-\eta_2)+\eta_1\eta_2}\left(\frac{1}{1+\eta_1\eta_2 x^2}\right) \\
&= \frac{\delta_{\eta_1\eta_2;1}}{2(1+x^2)} + \frac{\delta_{\eta_1\eta_2;-1}}{i(\eta_1-\eta_2)(1-x^2)}\ . \qquad (420)
\end{aligned}
$$

Let us now compute the second term in Eq. (419):

$$
\begin{aligned}
&{}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B^{(2)}_{\mathrm{sgh}}| x^{-2\gamma_0\beta_0} |B^{(2)}_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} \\
&= {}_{-\frac{3}{2}}\langle\tilde{0}| \otimes {}_{-\frac{1}{2}}\langle 0| \frac{e^{i\eta_2\tilde{\beta}_0\gamma_0}}{1+i\eta_2} x^{-2\gamma_0\beta_0} \frac{e^{-i\eta_1\tilde{\gamma}_0\beta_0}}{1-i\eta_1} |\tilde{0}\rangle_{-\frac{1}{2}} \otimes |0\rangle_{-\frac{3}{2}}\ . \qquad (421)
\end{aligned}
$$

By observing that:

$$
[\gamma_0\beta_0,\ \beta_0^n] = n\beta_0^n \qquad (422)
$$

and using the identity:

$$
e^A B e^{-A} = B + [A,\ B] + \frac{1}{2}[A,\ [A,\ B]] + \ldots \qquad (423)
$$

we can compute:

$$
x^{-2\gamma_0\beta_0}\beta_0^n x^{2\gamma_0\beta_0} = x^{-2n}\beta_0^n \qquad (424)
$$

from which it follows:

$$
x^{-2\gamma_0\beta_0} e^{-i\eta\tilde{\gamma}_0\beta_0} x^{2\gamma_0\beta_0} = e^{-i\eta x^{-2}\tilde{\gamma}_0\beta_0}\ . \qquad (425)
$$

Furthermore, since

$$
\gamma_0|0\rangle_{-\frac{3}{2}} = 0, \qquad (426)
$$

we can write:

$$
\begin{aligned}
&{}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B^{(2)}_{\mathrm{sgh}}| x^{-2\gamma_0\beta_0} |B^{(2)}_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} \\
&= \left(\frac{1}{x^2}\right) {}_{-\frac{3}{2}}\langle\tilde{0}| \otimes {}_{-\frac{1}{2}}\langle 0| \frac{e^{i\eta_2\tilde{\beta}_0\gamma_0}}{1+i\eta_2} x^{-2\gamma_0\beta_0} \frac{e^{-i\eta_1\tilde{\gamma}_0\beta_0}}{1-i\eta_1} x^{2\gamma_0\beta_0} |\tilde{0}\rangle_{-\frac{1}{2}} \otimes |0\rangle_{-\frac{3}{2}} \\
&= \left(\frac{1}{x^2}\right) {}_{-\frac{3}{2}}\langle\tilde{0}| \otimes {}_{-\frac{1}{2}}\langle 0| \frac{e^{i\eta_2\tilde{\beta}_0\gamma_0}}{1+i\eta_2} \frac{e^{-i\eta_1 x^{-2}\tilde{\gamma}_0\beta_0}}{1-i\eta_1} |\tilde{0}\rangle_{-\frac{1}{2}} \otimes |0\rangle_{-\frac{3}{2}} \\
&= \frac{1}{1-i(\eta_1-\eta_2)+\eta_1\eta_2}\left(\frac{1}{x^2+\eta_1\eta_2}\right) = \frac{\delta_{\eta_1\eta_2;1}}{2(1+x^2)} + \frac{\delta_{\eta_1\eta_2;-1}}{i(\eta_2-\eta_1)(x^2-1)}, \qquad (427)
\end{aligned}
$$

which concides with Eq. (420).

In conclusion we have:

$$ {}^{(0)}_{\mathrm{R-R}}\langle \eta_2, B_{\mathrm{sgh}}|x^{-2\gamma_0\beta_0}|B_{\mathrm{sgh}}, \eta_1\rangle^{(0)}_{\mathrm{R-R}} = \frac{\delta_{\eta_1\eta_2;1}}{2(1+x^2)} + \frac{\delta_{\eta_1\eta_2;-1}}{i(\eta_1-\eta_2)(1-x^2)}, \tag{428} $$

which coincides with the results given in Ref.s [85] and [86].

Finally, we give the action of the operator $I_{9-p'}$ with p' odd on the boundary state:

$$ \begin{aligned} & I_{9-p'}\left[C\Gamma^0\ldots\Gamma^{p'}(1+\eta\Gamma^{11})\right]_{AB}|A\rangle|\tilde{B}\rangle \\ & = \left[C\Gamma^0\ldots\Gamma^{p'}(1+\eta\Gamma^{11})\right]_{AB}\left(\Gamma^9\Gamma^8\cdots\Gamma^{p'+1}\right)^A{}_D\left(\Gamma^9\Gamma^8\cdots\Gamma^{p'+1}\right)^B{}_C|D\rangle|\tilde{C}\rangle \\ & = \left[\left(\Gamma^9\Gamma^8\cdots\Gamma^{p'+1}\right)^T C\Gamma^0\ldots\Gamma^{p'}(1+\eta\Gamma^{11})\Gamma^9\Gamma^8\cdots\Gamma^{p'+1}\right]_{AB}|A\rangle|\tilde{B}\rangle \\ & = \left[C\Gamma^0\ldots\Gamma^{p'}(1+\eta\Gamma^{11})\right]_{AB}|A\rangle|\tilde{B}\rangle \ . \end{aligned} \tag{429} $$

In the latter two equations we have used the following identities:

$$ \left(\Gamma^{11}\right)^T = -C\Gamma^{11}C^{-1} \qquad \left(\Gamma^M\right)^T = -C\Gamma^M C^{-1} \quad M = 0\ldots 9 \ . \tag{430} $$

C Euler-Heisenberg actions

We start by summarizing the calculation of the Euler-Heisenberg action for an arbitrary gauge theory containing N_s real scalars and N_f Dirac fermions and described by the following Lagrangian in d space-time dimensions:

$$ L = -\frac{1}{4g^2}F^a_{\mu\nu}F^a_{\mu\nu} + \frac{1}{2}(D_\mu\Phi)^i(D_\mu\Phi)^i + i\bar{\Psi}^i\gamma^\mu D^{ab}_\mu\Psi^i \ . \tag{431} $$

If we expand it around a background $\bar{A}^a_\mu$ solution of the classical equations of motion, assuming that the fluctuation $\mathcal{A}$ satisfies the background gauge condition,

$$ A^a_\mu = \bar{A}^a_\mu + \mathcal{A}^a_\mu \ , \ (\bar{D}_\mu\mathcal{A}_\mu)^a = 0 \tag{432} $$

and keeping only up to the quadratic terms in the fluctuations, we get the Euler-Heisenberg effective action:

$$ S_{EH} = \frac{1}{2}Tr\log\Delta_1 + (\frac{N_s}{2}-1)Tr\log\Delta_0 - N_f Tr\log\Delta_{1/2} \ , \tag{433} $$

where we have neglected the contribution of the classical action, and

$$ (\Delta_1)^{ab}_{\mu\nu} = -(\bar{D}^2)^{ab}\delta_{\mu\nu} + 2f^{acb}\bar{F}^c_{\mu\nu} \quad (\Delta_0)^{ij} = -(\bar{D}^2)^{ij} \quad (\Delta_{\frac{1}{2}})^{ij} = i\gamma^\mu(\bar{D}_\mu)^{ij}. \tag{434} $$

$\bar{D}_\mu$ is the covariant derivative computed in the classical background. The determinant in Eq. (433) can be explicitly computed if we assume that the field strength corresponding to the background $\bar{A}$ is constant. In this case we get:

$$ S_{EH} = NI_1 + c_s I_0 + c_f I_{1/2} \ , \tag{435} $$

where

$$I_0 = \frac{1}{(4\pi)^{d/2}} \int d^d x \int_0^\infty \frac{d\sigma}{\sigma^{1+d/2}} e^{-\sigma m^2} \cdot \frac{\hat{f}\sigma}{\sin(\hat{f}\sigma)} \cdot \frac{\hat{g}\sigma}{\sinh(\hat{g}\sigma)} \tag{436}$$

for the scalar,

$$I_1 = \frac{N}{(4\pi)^{d/2}} \int d^d x \int_0^\infty \frac{d\sigma}{\sigma^{1+d/2}} e^{-\sigma m^2} \times$$

$$\times \frac{\hat{f}\sigma}{\sin(\hat{f}\sigma)} \cdot \frac{\hat{g}\sigma}{\sinh(\hat{g}\sigma)} \left[d - 2 + 4\sinh^2(\hat{g}\sigma) - 4\sin^2(\hat{f}\sigma) \right] \tag{437}$$

for the gluon that includes also the contribution of the Faddev-Popov ghost and

$$I_{1/2} = -\frac{2^{[d/2]}}{(4\pi)^{d/2}} \int d^d x \int_0^\infty \frac{d\sigma}{\sigma^{1+d/2}} e^{-\sigma m^2} \frac{\hat{f}\sigma}{\sin(\hat{f}\sigma)} \frac{\hat{g}\sigma}{\sinh(\hat{g}\sigma)} cos(\hat{f}\sigma) cosh(\hat{g}\sigma) \tag{438}$$

for a complex fermion. We have introduced an infrared cut-off m and we have taken the constant field strength with the following form:

$$\bar{F}_{\alpha\beta} = \begin{pmatrix} 0 & \hat{f} & 0 & 0 & \cdot \\ -\hat{f} & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & \hat{g} & \cdot \\ 0 & 0 & -\hat{g} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad . \tag{439}$$

All the other matrix elements are zero. The constants c_s and c_f set the normalization of the generators of the $SU(N)$ gauge group, namely $Tr(T^A T^B) = c\delta^{AB}$ and $[\frac{d}{2}] = \frac{d}{2}$ if d is even and $[\frac{d}{2}] = \frac{d-1}{2}$ if d is odd. Remember that $c = N$ in the adjoint representation of $SU(N)$ as we have used in Eq. (435) for the gluon contribution. Eq.s (436) and (438) are derived in Appendix B of Ref. [89] and Eq. (437) can be derived in a similar way.

In the following we perform the field theory limit of the various one-loop open string contributions with a constant gauge field that we constructed for different models. By this limit we mean:

$$\tau \to \infty \ \ , \ \ \alpha' \to 0 \ \ ; \ \ \sigma \equiv 2\pi\alpha'\tau \ , \ \hat{f} \ , \ \hat{g} \ fixed. \tag{440}$$

We start by considering the bosonic string. In this case the annulus diagram with a constant gauge field is given in Eq.s (91) and (92). In the case of the bosonic string the field theory limit is not well-defined because we have an open string tachyon and we have to eliminate its contribution by hand keeping only the contribution of the massless open string states. In the field theory limit we can use the expressions:

$$\nu_f \sim -2i\alpha'\hat{f} \quad , \quad \nu_g \sim -2\alpha'\hat{g} \ \ , \tag{441}$$

$$\Theta_1(i\nu_f\tau|i\tau) \sim -2k^{1/6} f_1(k) \sin(\hat{f}\sigma) \left[1 - 2\cos(2\hat{f}\sigma)k^2 + \dots \right] \tag{442}$$

and

$$\Theta_1(i\nu_g\tau|i\tau) \sim 2ik^{1/6}f_1(k)\sinh(\hat{g}\sigma)\left[1-2\cosh(2\hat{g}\sigma)k^2+\ldots\right] \quad , \tag{443}$$

where the dots denote the contribution of the massive states. By using the previous equations together with Eq. (402), we get for the untwisted contribution in Eq. (91) the following expression:

$$(Z_e^o)_{bos}^{ftl} = \frac{N}{2(4\pi)^2}\int d^4x \int_0^\infty \frac{d\sigma}{\sigma^3}e^{-m^2\sigma}\frac{\hat{f}\sigma}{\sin\hat{f}\sigma}\frac{\hat{g}\sigma}{\sinh\hat{g}\sigma}$$
$$\times\left[24-4\sin^2(\hat{f}\sigma)+4\sinh^2(\hat{g}\sigma)\right] \quad , \tag{444}$$

where $m \equiv \frac{y}{2\pi\alpha'}$. If we remember that in our case we are considering a D3 brane in a 26-dimensional space, we see that the previous equation, multiplied by a factor 2 due to the missing orbifold projector, corresponds to the four-dimensional Euler-Heisenberg action for a gluon and 22 adjoint scalars:

$$(Z_e^o)_{bos}^{ftl} = N(I_1 + 22I_0) \ . \tag{445}$$

The field theory limit of the twisted contribution can be performed in the same way. When we sum it to the untwisted one we get:

$$(Z^{ftl})_{bos} = N(I_1 + N_sI_0) \quad , \tag{446}$$

that is equal to the Euler-Heisemberg action for a gluon and N_s scalar in the adjoint representation of $SU(N)$.

Let us now perform the field theory limit on the sum of the untwisted sector in Eq. (40) and of the twisted sector one in Eq. (41). We have to use the following equations valid in the field theory limit:

$$\Theta_3(i\nu\tau|i\tau) \sim 1+2\cosh(2\pi\nu\tau)k \quad , \quad \Theta_4(i\nu\tau|i\tau) \sim 1-2\cosh(2\pi\nu\tau)k \tag{447}$$

$$\Theta_1(i\nu\tau|i\tau) \sim -2k^{1/4}\sin(i\pi\nu\tau) \quad , \quad \Theta_2(i\nu\tau|i\tau) \sim 2k^{1/4}\cos(i\pi\nu\tau) \quad . \tag{448}$$

In particular the field theory limit is given by:

$$Z^o \equiv -\frac{N}{(4\pi)^2}\int d^4x \int_0^\infty \frac{d\sigma}{\sigma}e^{-m^2\sigma}\frac{\hat{f}\sigma}{\sin(\hat{f}\sigma)}\cdot\frac{\hat{g}\sigma}{\sinh(\hat{g}\sigma)}\left(\hat{Z}_e^o+\hat{Z}_h^o\right) \quad , \tag{449}$$

where:

$$\begin{aligned}
\hat{Z}_e^o &\simeq -\frac{1}{4k}\Big[(1+4k)(1+2\cos(2\hat{f}\sigma)k+2\cosh(2\hat{g}\sigma)k) \\
&- (1-4k)(1-2\cos(2\hat{f}\sigma)k-2\cosh(2\hat{g}\sigma)k)-16k\cos(\hat{f}\sigma)\cosh(\hat{g}\sigma)\Big] \\
&= -\frac{1}{2}\left[\left(8+4\sin^2(\hat{f}\sigma)-4\sinh(\hat{g}\sigma)\right)-8\cos(\hat{f}\sigma)\cosh(\hat{g}\sigma)\right] && (450) \\
\hat{Z}_h^o &\simeq 2\left[\sin^2(\hat{f}\sigma)-\sinh^2(\hat{g}\sigma)\right] \ . && (451)
\end{aligned}$$

This means that the contribution of the untwisted sector multiplied by a factor 2, in order to get rid of the factor 1/2 of the orbifold projection, is equal to:

$$(Z_e^o)^{ftl} = \frac{N}{(4\pi)^2} \int d^4x \int_0^\infty \frac{d\sigma}{\sigma^3} e^{-m^2\sigma} \frac{\hat{f}\sigma}{\sin(\hat{f}\sigma)} \cdot \frac{\hat{g}\sigma}{\sinh(\hat{g}\sigma)} \times$$
$$\times \left[8 + 4\sinh^2(\hat{g}\sigma) - 4\sin^2(\hat{f}\sigma) - 8\cos(\hat{f}\sigma)\cosh(\hat{g}\sigma)\right] \quad , \tag{452}$$

where the first three terms in the last line come from the NS sector and correspond to one gluon and six scalars, while the last term comes from the R sector and corresponds to 4 Majorana fermions. Both the scalars and the fermions are in the adjoint representation of $SU(N)$. In conclusion the previous equation can be written as:

$$(Z_e^o)^{ftl} = N(I_1 + 6I_0 + 2I_{1/2}) \quad , \tag{453}$$

that is the Euler-Heisenberg action of $\mathcal{N} = 4$ super Yang-Mills. It is also equal to what one gets for a D3 brane in type 0' theory because the contribution of the Möbius diagram is vanishing for a D3 brane (see Eq.s (244) and (253)).

The field theory limit of the twisted contribution in Eq. (41) is given by:

$$(Z_h^o)^{ftl} = -\frac{N}{(4\pi)^2} \int d^4x \int_0^\infty \frac{d\sigma}{\sigma^3} e^{-m^2\sigma} \frac{\hat{f}\sigma}{\sin(\hat{f}\sigma)} \frac{\hat{g}\sigma}{\sinh(\hat{g}\sigma)}$$
$$\times \left[2\sin^2(\hat{f}\sigma) - 2\sinh^2(\hat{g}\sigma)\right] \quad , \tag{454}$$

with in addition the θ term that we omit to write here.

Multiplying the contribution in Eq. (452) with a factor 1/2 due to the orbifold projection and summing it to the twisted one in Eq. (454), we get the Euler-Heisenberg action of $\mathcal{N} = 2$ super Yang-Mills:

$$Z^{ftl}_{\mathcal{N}=2} = N(I_1 + 2I_0 + I_{1/2}) \quad . \tag{455}$$

In the last part of this Appendix we perform the field theory limit of the orientifold $\Omega' I_6$ discussed in Sect. (7.3). The annulus diagram for a D3 brane of this orientifold is equal to the untwisted part of the annulus diagram for a D3 brane of type IIB theory on the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. In the field theory limit one gets the expression in Eq. (452).

The contribution of the Möbius diagram to the Euler-Heisenberg action can be obtained from Eq. (329) using in the field theory limit (see Eq.(440)) Eq.s (399)÷(402). We get:

$$Z^F \simeq \mp \frac{16}{(4\,\pi)^2} \int d^4x \int_0^\infty \frac{d\sigma}{\sigma} \mathrm{e}^{-\frac{y^2}{(2\pi\alpha')^2}\sigma}\; \hat{f}\,\hat{g}\; \frac{\cos(\sigma\hat{f})\cosh(\sigma\hat{g})}{\sin(\sigma\hat{f})\sinh(\sigma\hat{g})} \quad . \tag{456}$$

Adding Eq.s (452) and (456) we get the total contribution for the theory described by the orientifold $\Omega' I_6$ that is equal to:

$$S^{or}_{EU} = N(I_1 + 6I_0) + 4\frac{N \pm 2}{2} I_{1/2} \quad , \tag{457}$$

that is the correct Euler-Heisenberg action for a system of one gluon, six adjoint scalars, but not the correct one for four Dirac fermions transforming according to the two-index (anti)symmetric representation of $SU(N)$.